

# A *Bicriterion* Concentration Inequality and Prophet Inequalities for *k-Fold Matroid Unions*

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# Outline

**Theorem:**  $\forall s \in [0,1], t > 0$

$$\Pr[f(\mathbf{X}^{(s)}) \geq \mathbf{E}[f(\mathbf{X})] + t] \leq e^{-st}$$

“Chernoff-strength” *bicriterion* concentration

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*Large girth* does not suffice for  $(1 - \varepsilon)$ -prophet inequality

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**Theorem:** There is  $(1 - O(\sqrt{\frac{\log k}{k}}))$ -competitive algorithm for any  **$k$ -fold matroid union  $\mathcal{F}^k$**

But  **$k$ -fold matroid unions** do

# Part I:

## A *Bicriterion* Concentration Inequality and Prophet Inequalities for *k-Fold Matroid Unions*



# Concentration inequalities

$n$  independent

Bernoulli r.v.s

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change output by  $\leq 1$ .”

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Focus on **upper tail** & **small deviation**  $t < \mu$

$$\Pr[f(\mathbf{X}) \geq \mu + t] \leq ?$$

# Example: Chernoff bound

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Implies standard deviation

$$\sigma = O(\sqrt{\mu})$$

# Example: submodular functions

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$$X = (X_1, X_2, \dots, X_n)$$

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Concentration for *self-bounding functions*  
[Boucheron/Lugosi/Massart '00]

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In general, impossible to get  
“Chernoff-strength” bound

- $\exists$  1-Lipschitz  $f$  such that  $f(\mathbf{X})$ :
- (Small expectation)  $\mu \ll \sqrt{n}$
  - (Large deviation)  $\sigma = \sqrt{n}$

Implies standard deviation  
 $\sigma = O(\sqrt{n})$

# Result 0: a *bicriterion* concentration

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set each  $X_i$  to 0 w.p.  $1 - e^{-s}$

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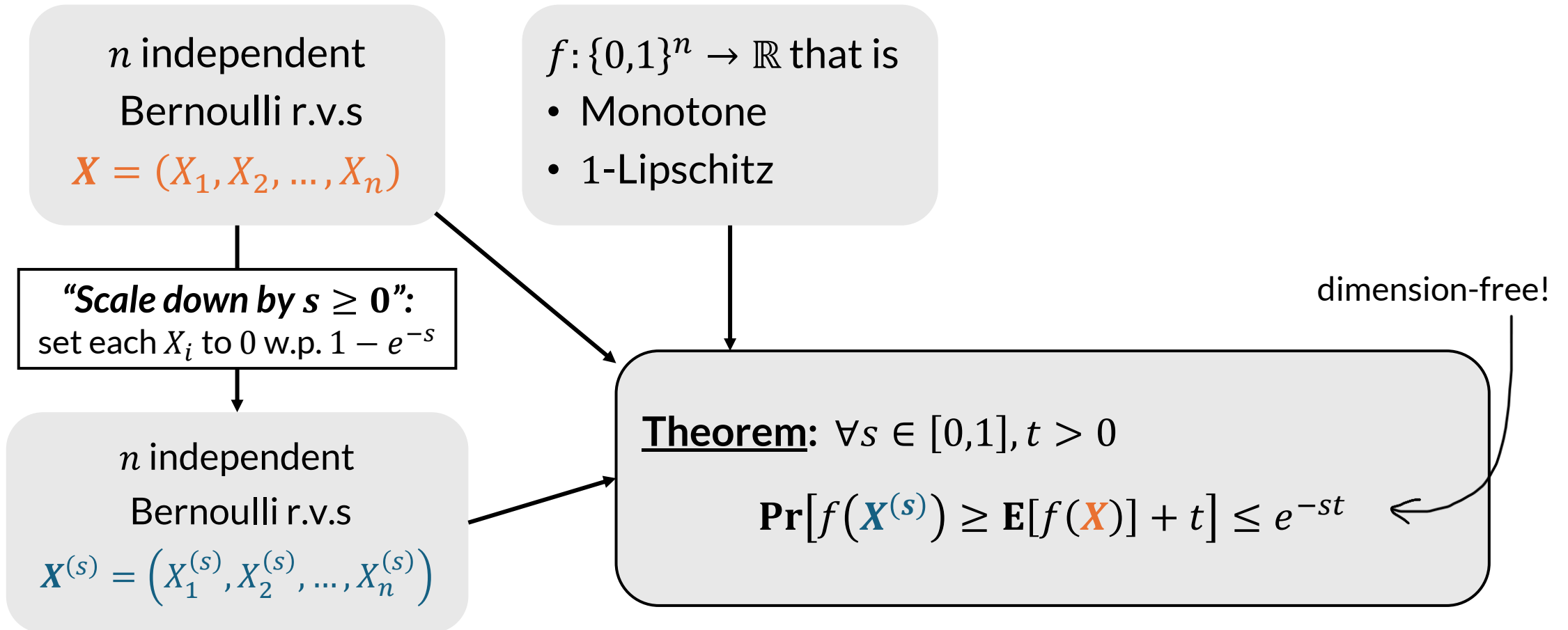
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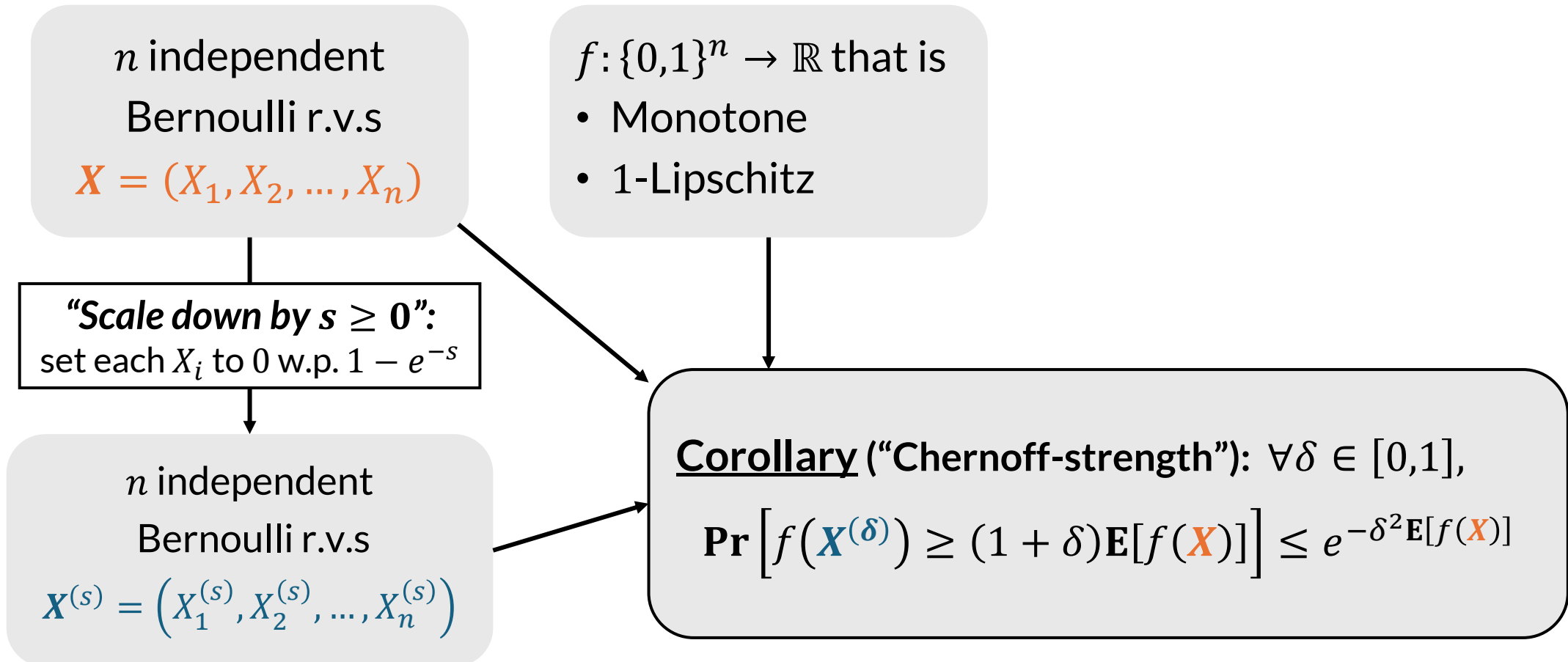
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**Intuition:** for  $f(\mathbf{x})$ , either:

- Changes **fast**:  
“Scaling-down” to  $f(\mathbf{X}^{(s)})$  helps
- Changes **slow**:  
 $f(\mathbf{X})$  already concentrates

**Corollary** (“Chernoff-strength”):  $\forall \delta \in [0,1]$ ,

$$\Pr \left[ f(\mathbf{X}^{(\delta)}) \geq (1 + \delta) \mathbf{E}[f(\mathbf{X})] \right] \leq e^{-\delta^2 \mathbf{E}[f(\mathbf{X})]}$$

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**Proof:** entropy method with

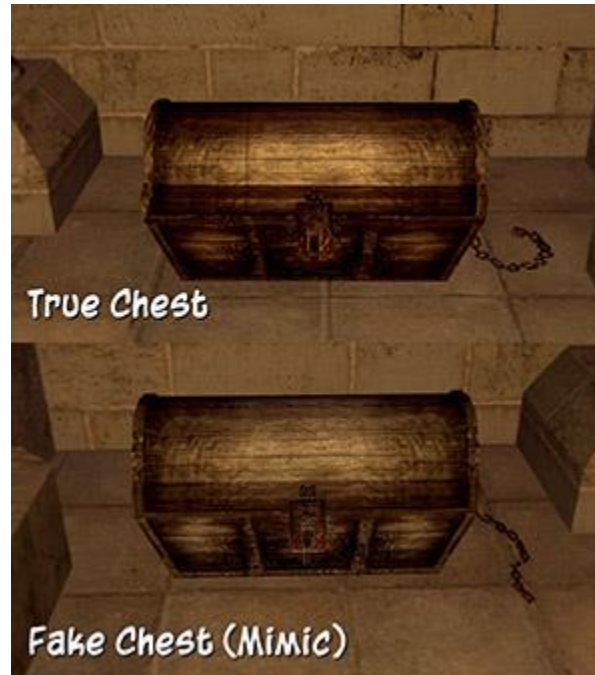
$$F(\lambda) = \mathbf{E} \left[ e^{-\lambda f(\mathbf{X}^{(\lambda)})} \right]$$

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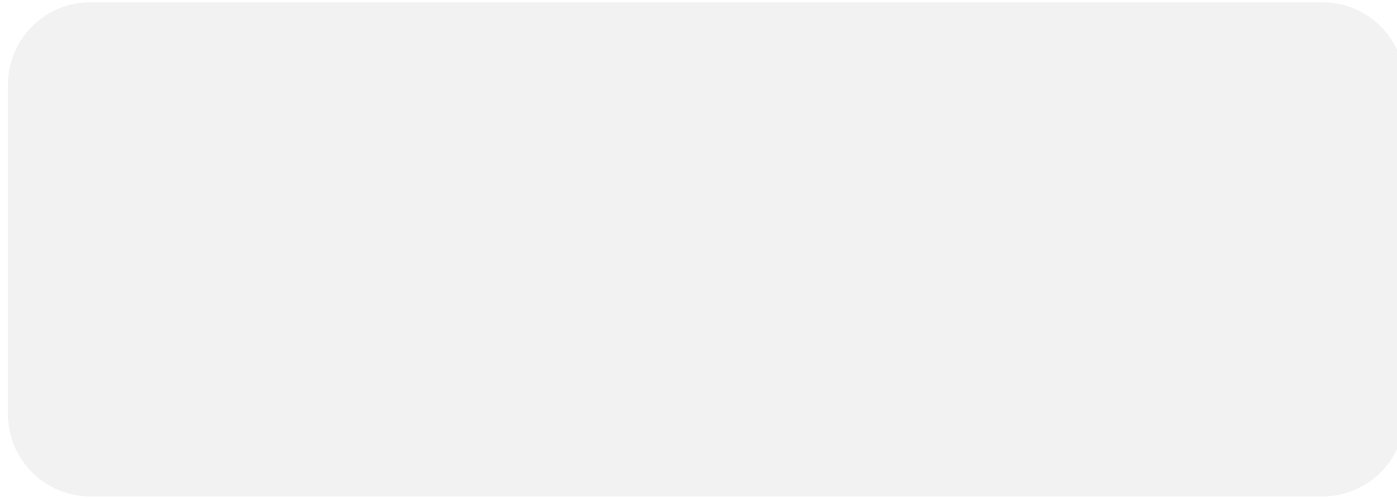
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## Part II:

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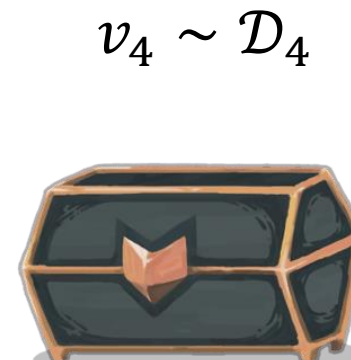
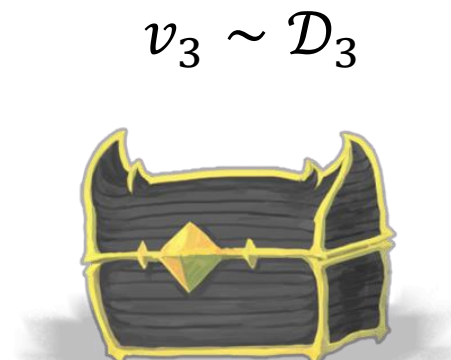
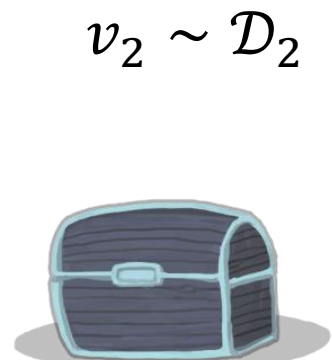


# Prophet inequalities



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  - **Accept**/reject  $v_i$  immediately and irrevocably

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Accept at most 1 item

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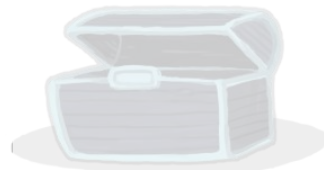
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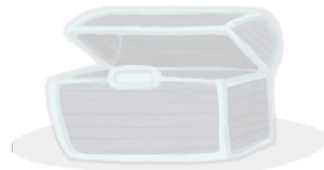
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  - vs. a *prophet* who gets  $\mathbf{E}[\max_i v_i]$

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**Prophet's value**  
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$\alpha$ -competitive:

$$\frac{\mathbf{E}[\text{ALG}]}{\mathbf{E}[\text{Prophet}]} \geq \alpha$$

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$\frac{1}{2}$ -competitive strategy: [Krengel/Sucheston/Garling '78, Samuel-Cahn '84]

**Accept** first  $v_i > T = \mathbf{Median}[\max_i v_i]$

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Tight in worst case

# Prophet inequalities (general feasibility)

- Given  $n$  independent  $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$ , **feasible sets**  $\mathcal{F} \subseteq 2^{[n]}$
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downward-closed



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Keep accepted items **feasible**



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  - Goal: maximize **sum of accepted values** in expectation
    - vs. a *prophet* who gets  $\mathbf{E}[\max_{S \in \mathcal{F}} \sum_{i \in S} v_i]$
- downward-closed
- Keep accepted items **feasible**

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- Because of a **large girth**?
- Because it is a ***union of many matroids***?

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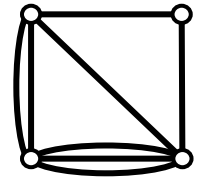
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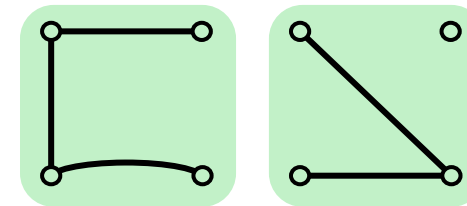
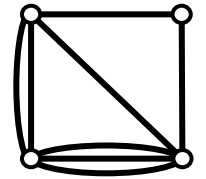
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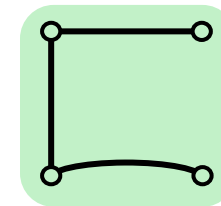
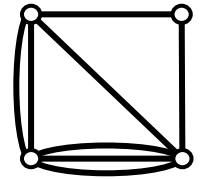
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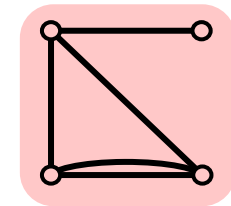
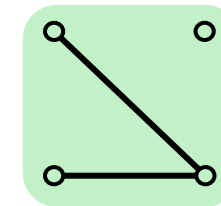
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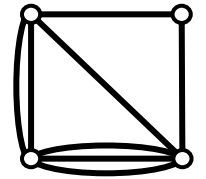
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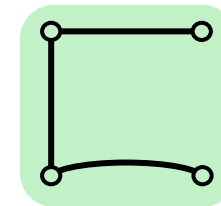
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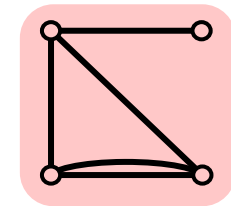
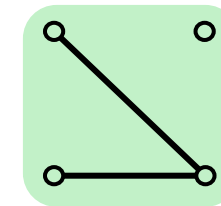
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**Theorem:**  $\forall k, \varepsilon$ , no  $(\frac{1}{2} + \varepsilon)$ -competitive algorithm for a graphical matroid  $\mathcal{F}_{k,\varepsilon}$  of *girth*  $k$



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Infeasible

# Result 2: *k*-fold matroid unions suffice

*k*-fold union of matroid  $\mathcal{F}$ :

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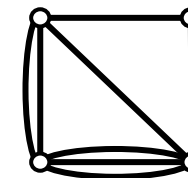
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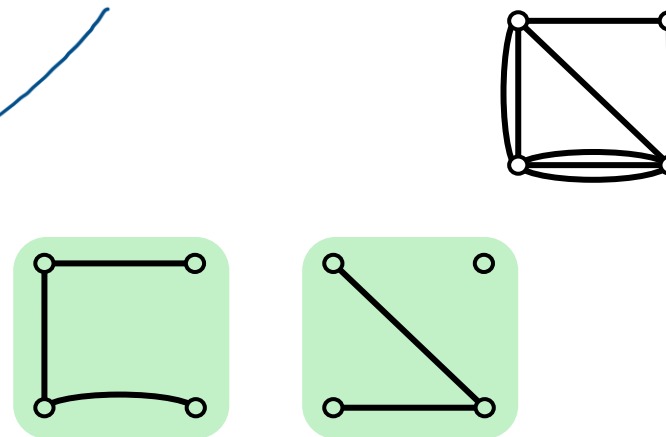
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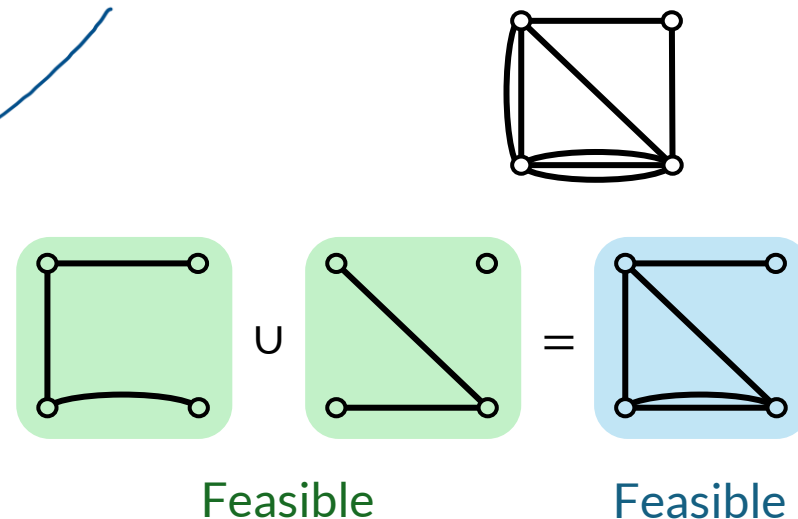
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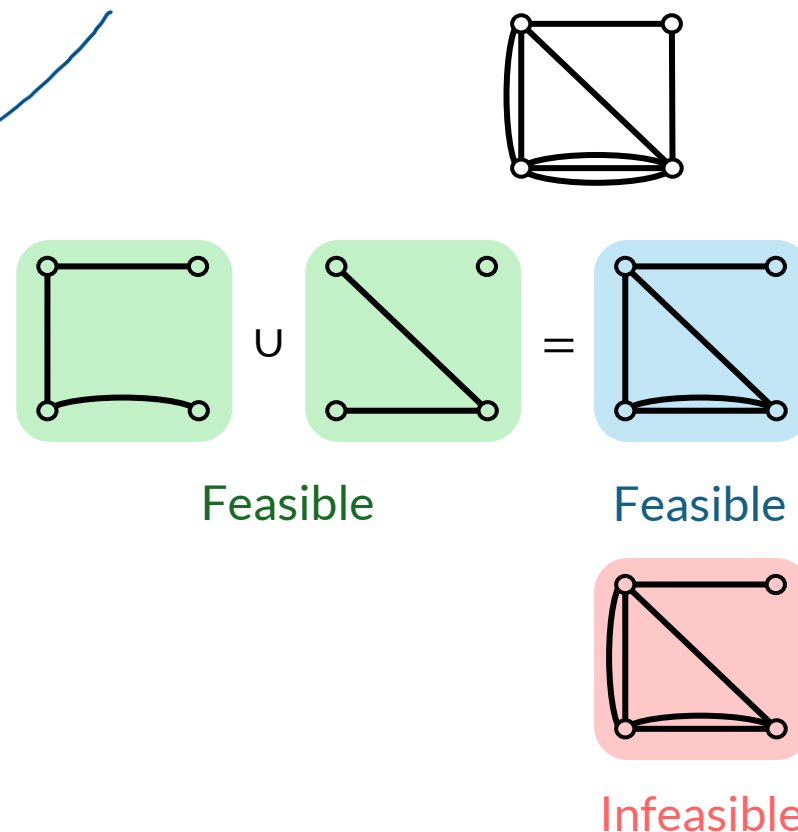
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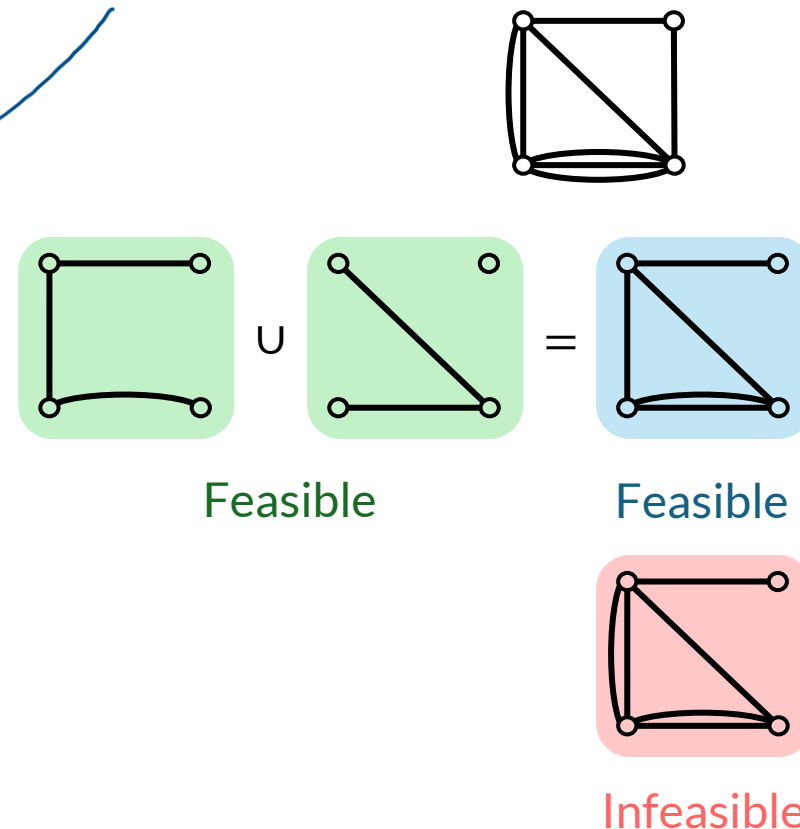
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**Theorem:** There is  $(1 - O(\sqrt{\frac{\log k}{k}}))$ -competitive algorithm for any  $k$ -fold matroid union  $\mathcal{F}^k$

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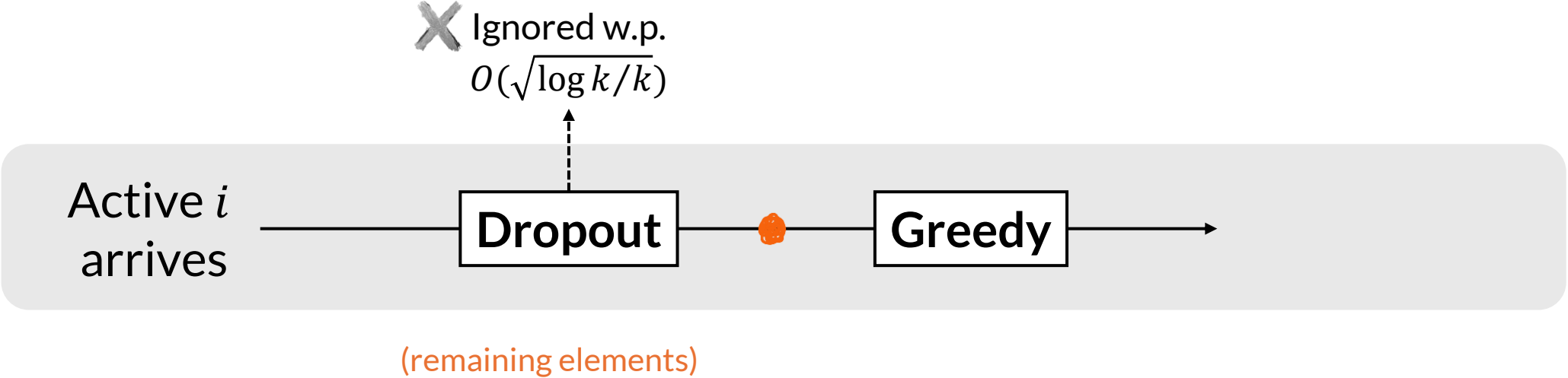
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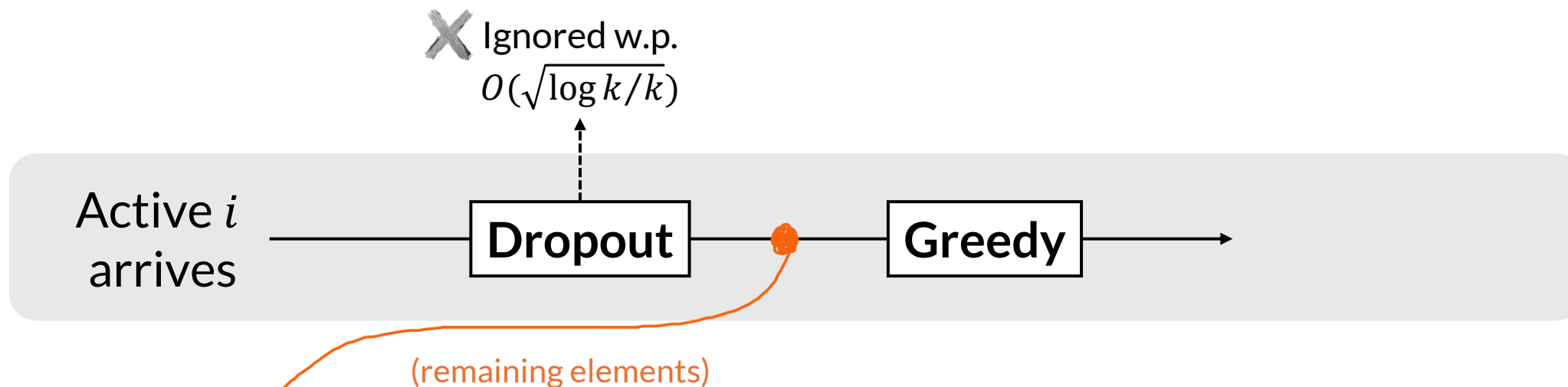
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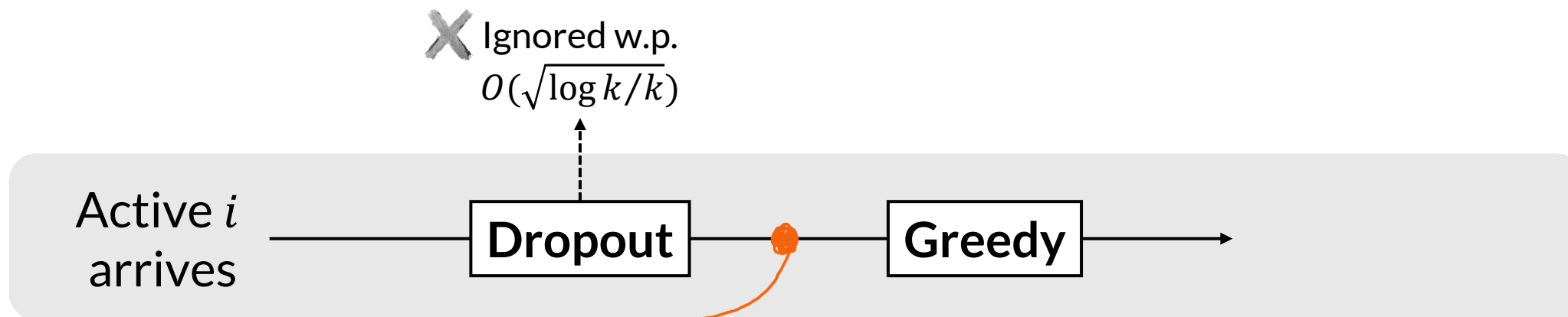
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Modest *occupancy of  $S$*  in expectation:

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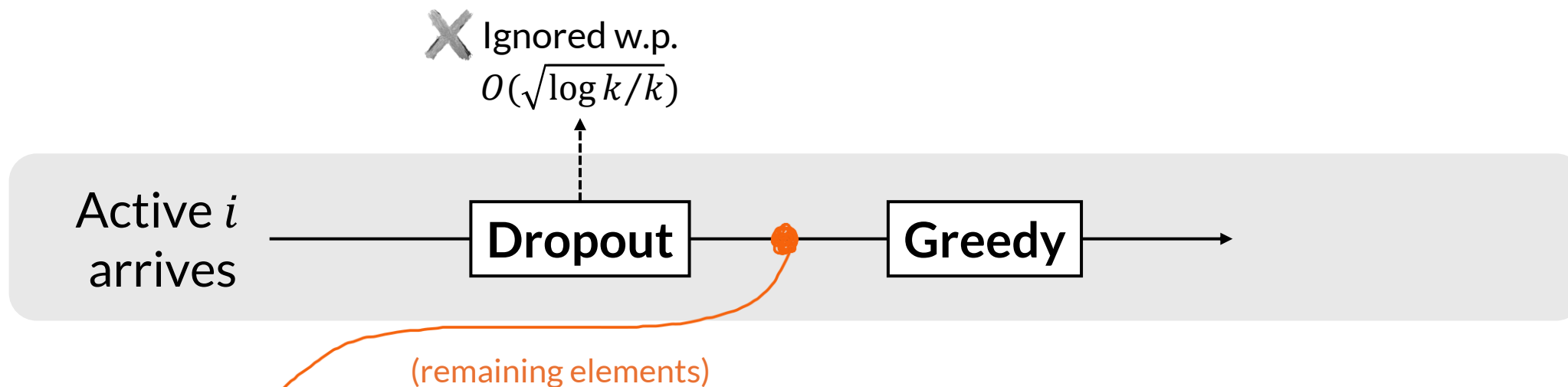
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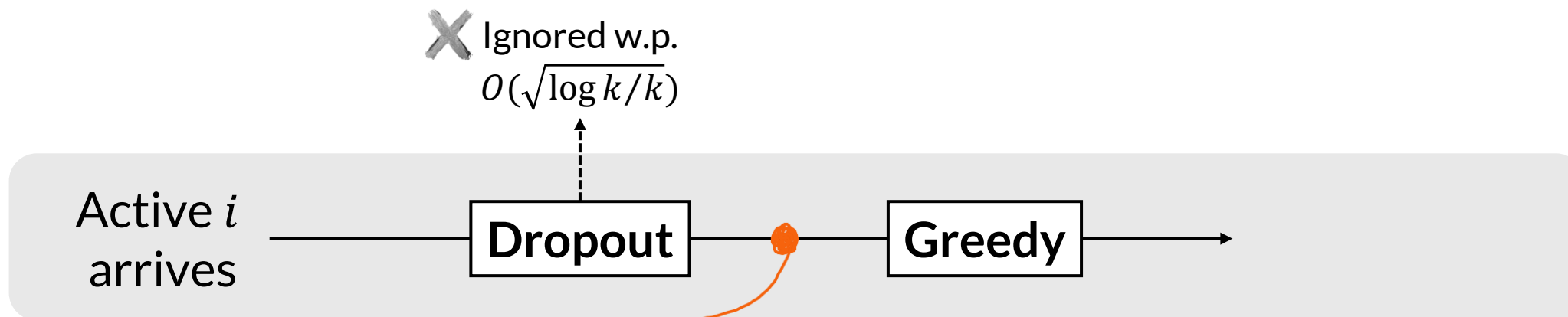
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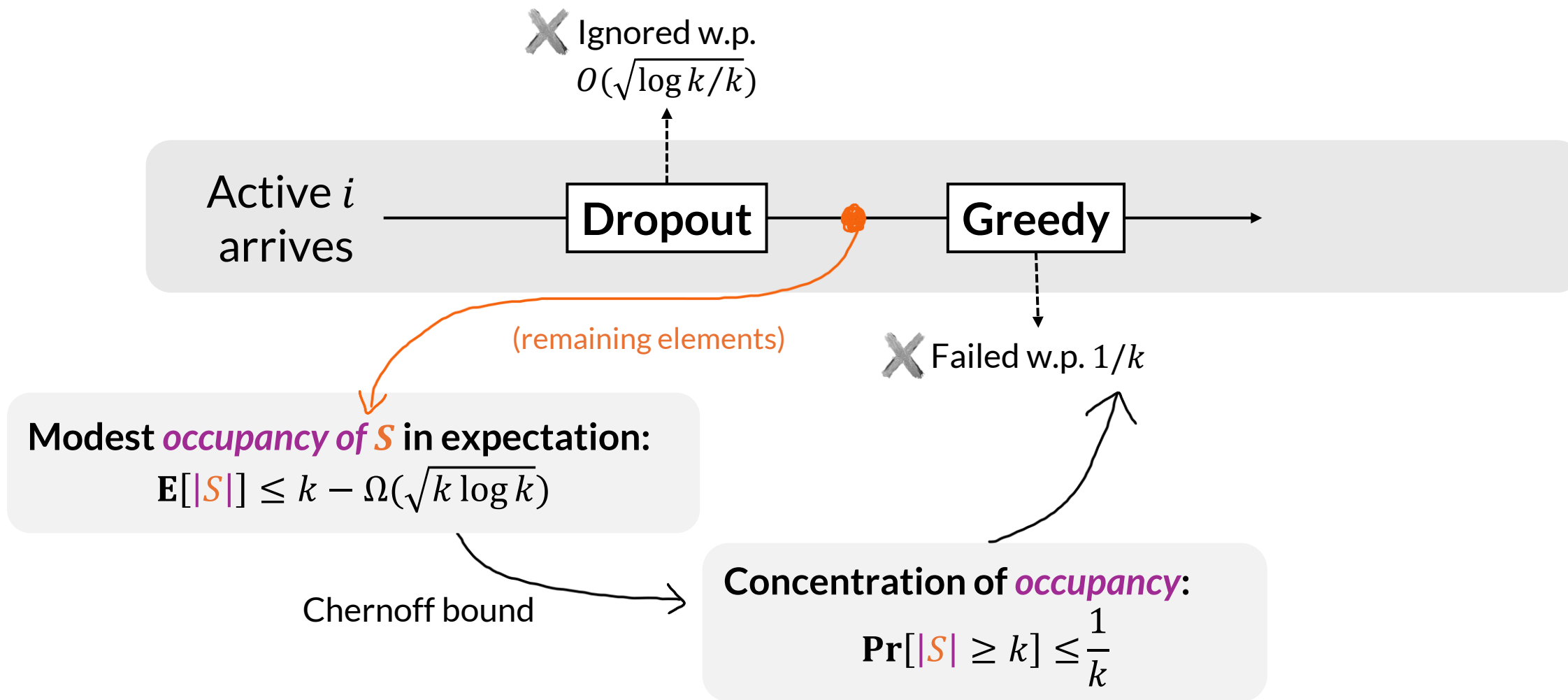
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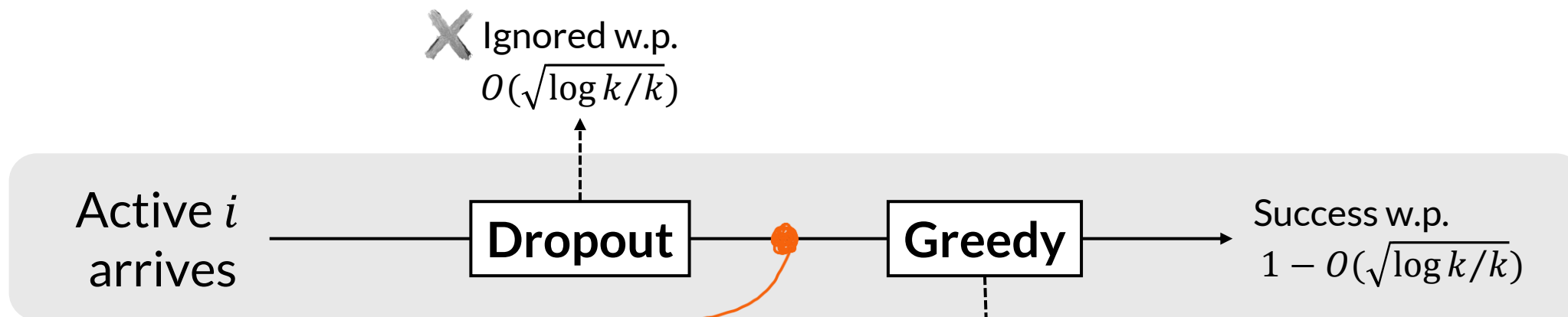
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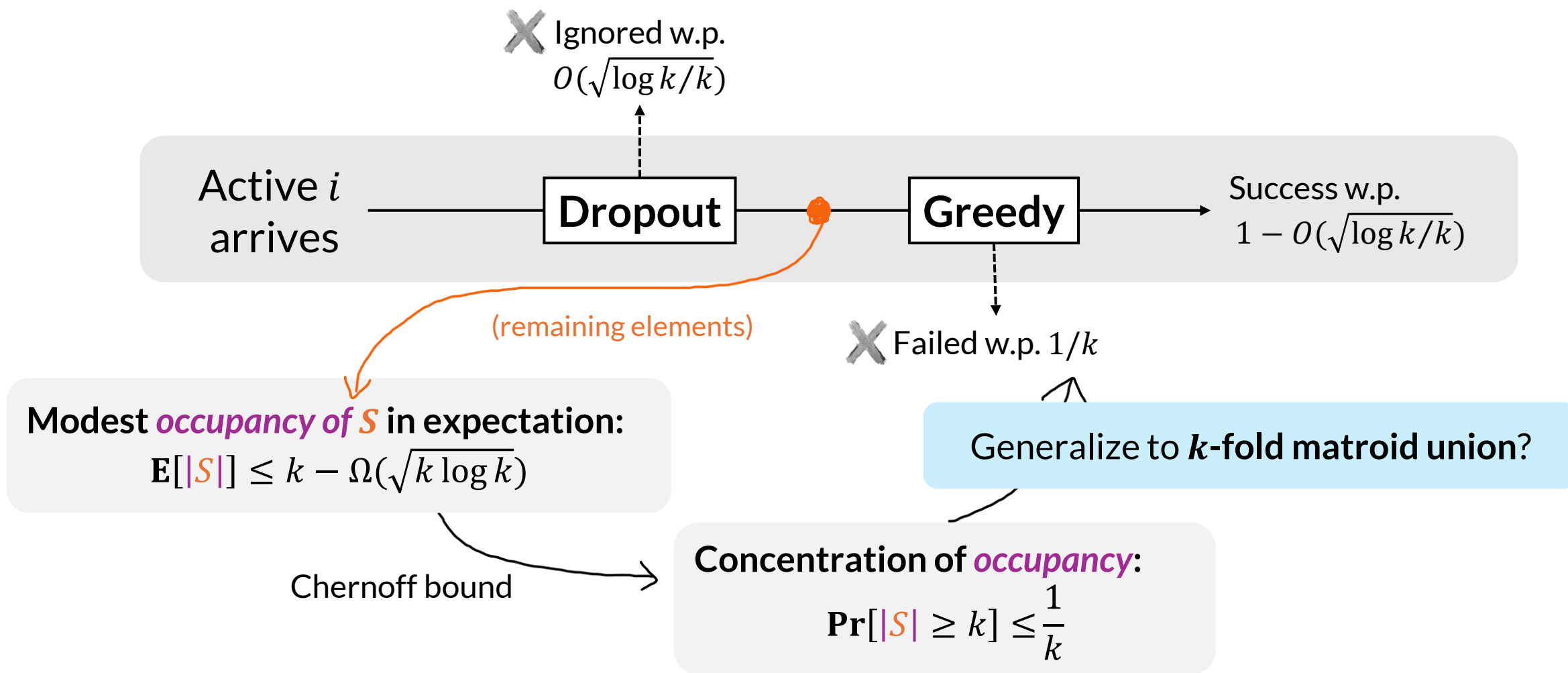
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*Occupancy*: “slots” occupied by  $S$

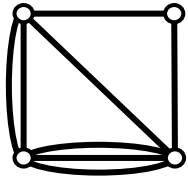
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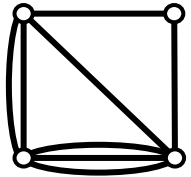
In 2-fold union of



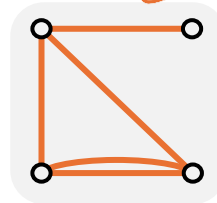
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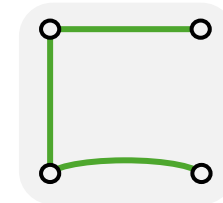
In 2-fold union of



Occupancy of

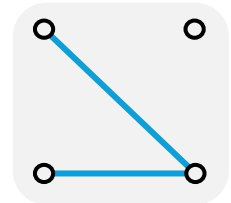


w.r.t.



Slot 1

$\cup$

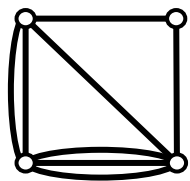


Slot 2

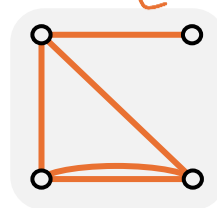
# Occupancy in $k$ -fold matroid unions

**Occupancy:** “slots” occupied by  $S$  with respect to specific element

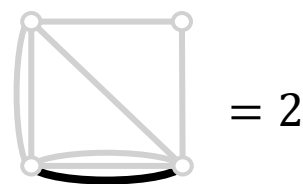
In 2-fold union of



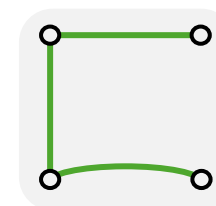
Occupancy of



w.r.t.

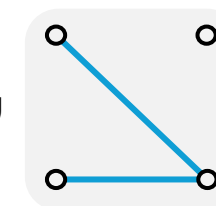


= 2



Slot 1

U

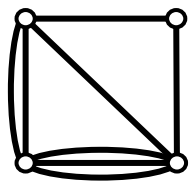


Slot 2

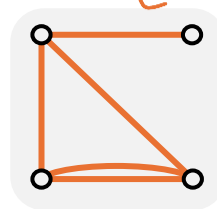
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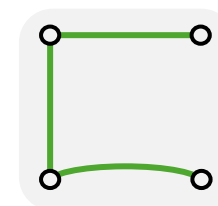
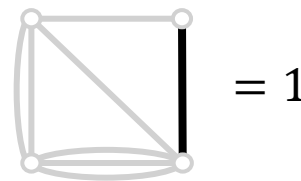
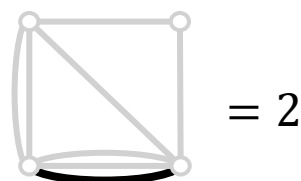
In 2-fold union of



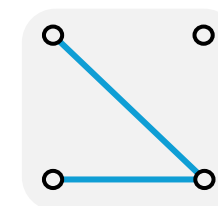
Occupancy of



w.r.t.



U

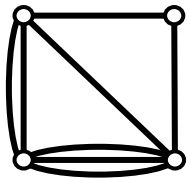




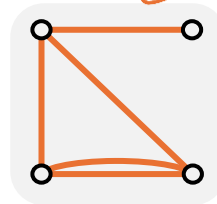
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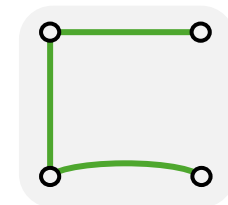
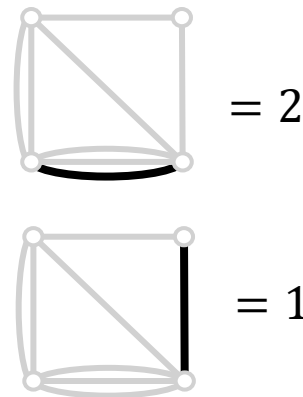
In 2-fold union of



Occupancy of

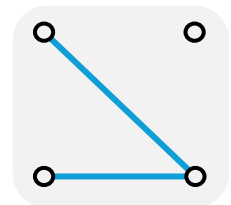


w.r.t.



Slot 1

U



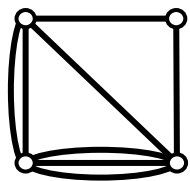
Slot 2

**Lemma:** there exists *occupancy function*  $\omega_i$  for every  $i$ :

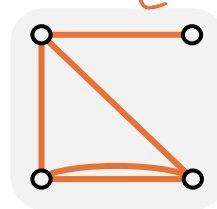
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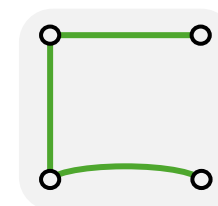
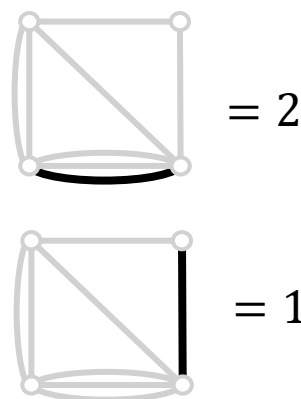
In 2-fold union of



Occupancy of

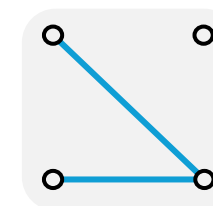


w.r.t.



Slot 1

U



Slot 2

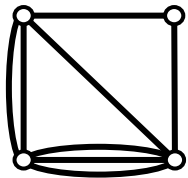
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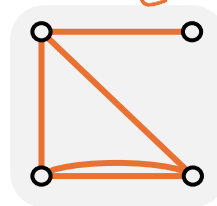
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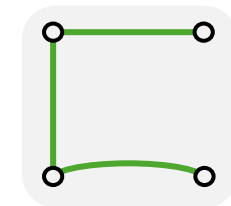
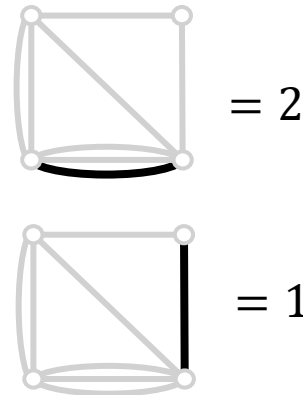
In 2-fold union of



Occupancy of

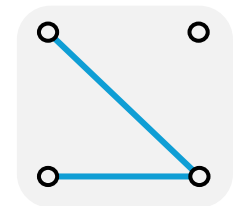


w.r.t.



Slot 1

U



Slot 2

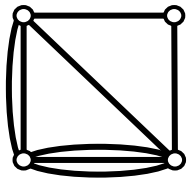
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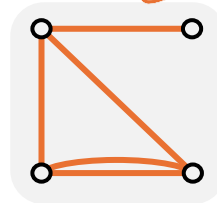
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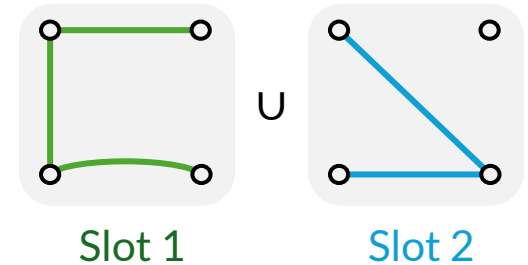
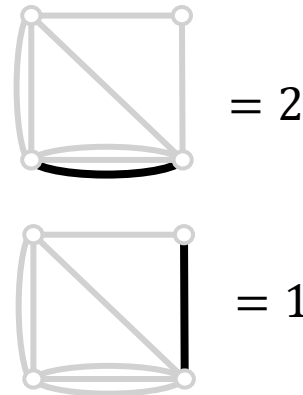
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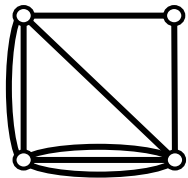
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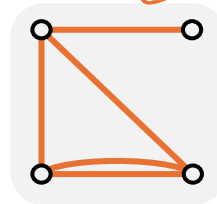
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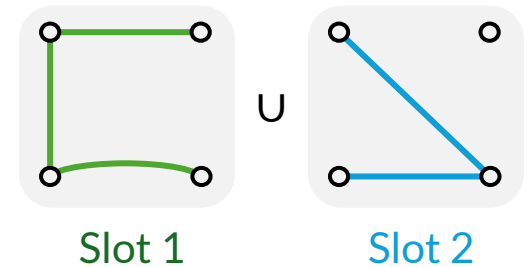
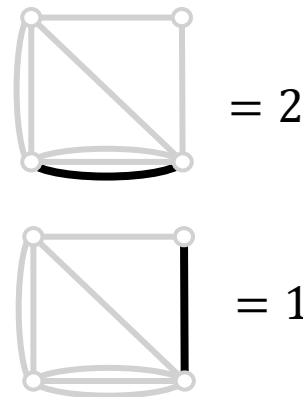
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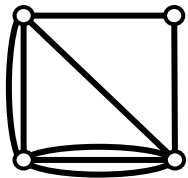
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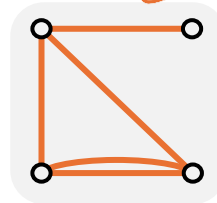
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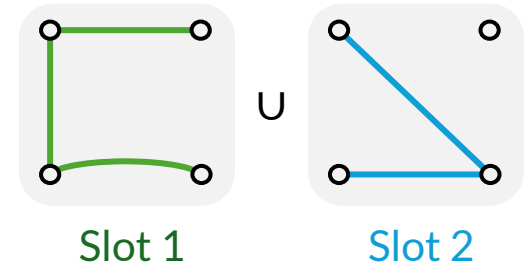
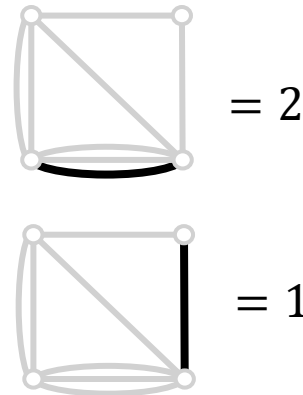
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Can be very general  
No hope for Chernoff-like bounds

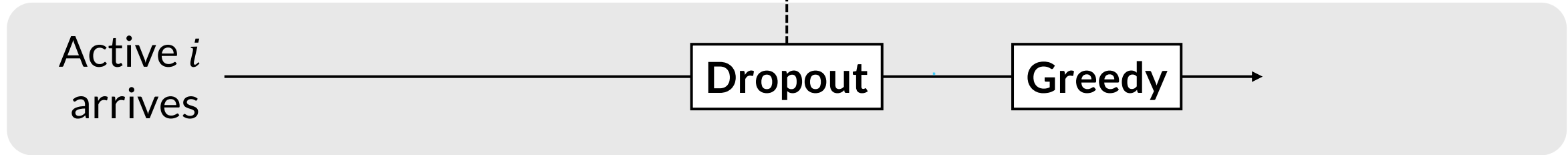
# Algorithm for $k$ -fold matroid unions

X Ignored w.p.  
 $O(\sqrt{\log k/k})$

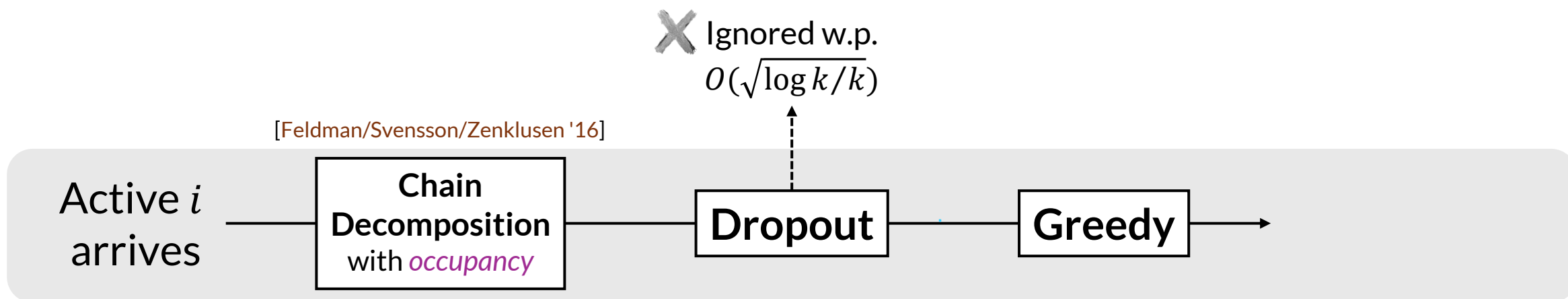
Active  $i$   
arrives

Dropout

Greedy

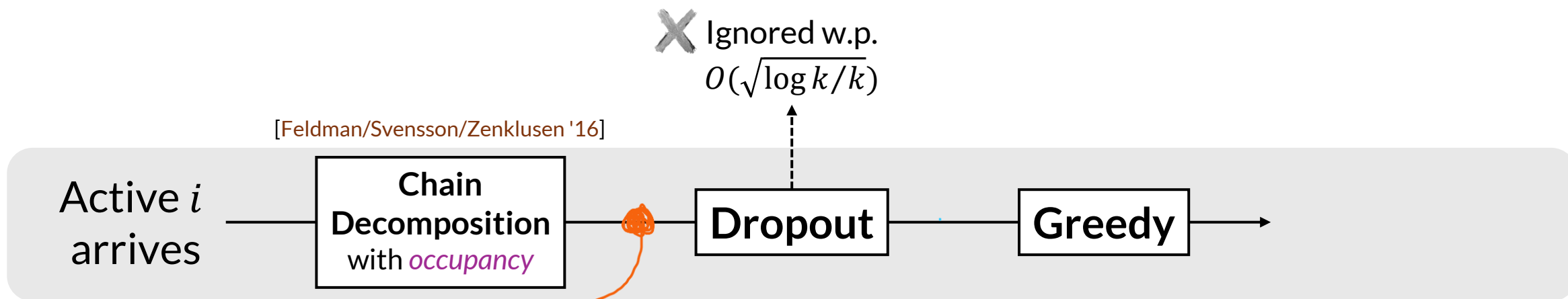


# Algorithm for $k$ -fold matroid unions





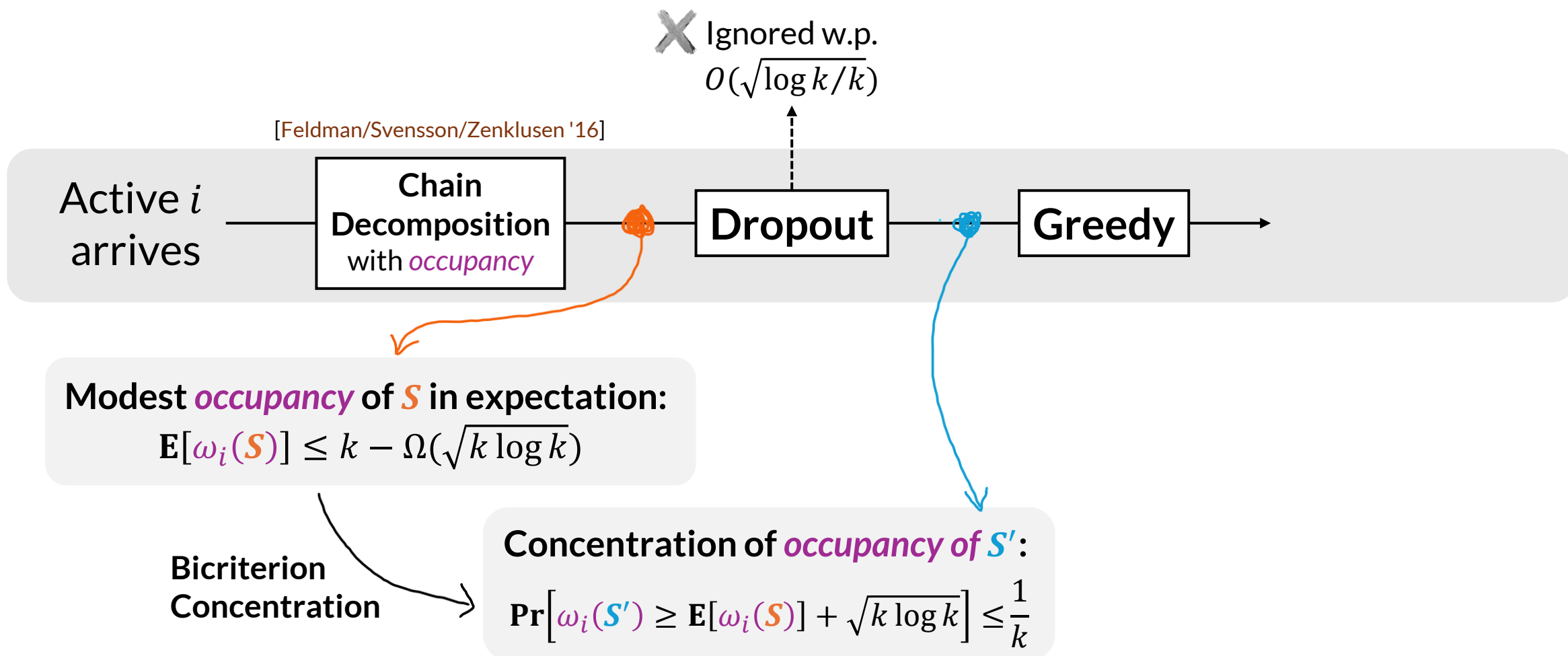
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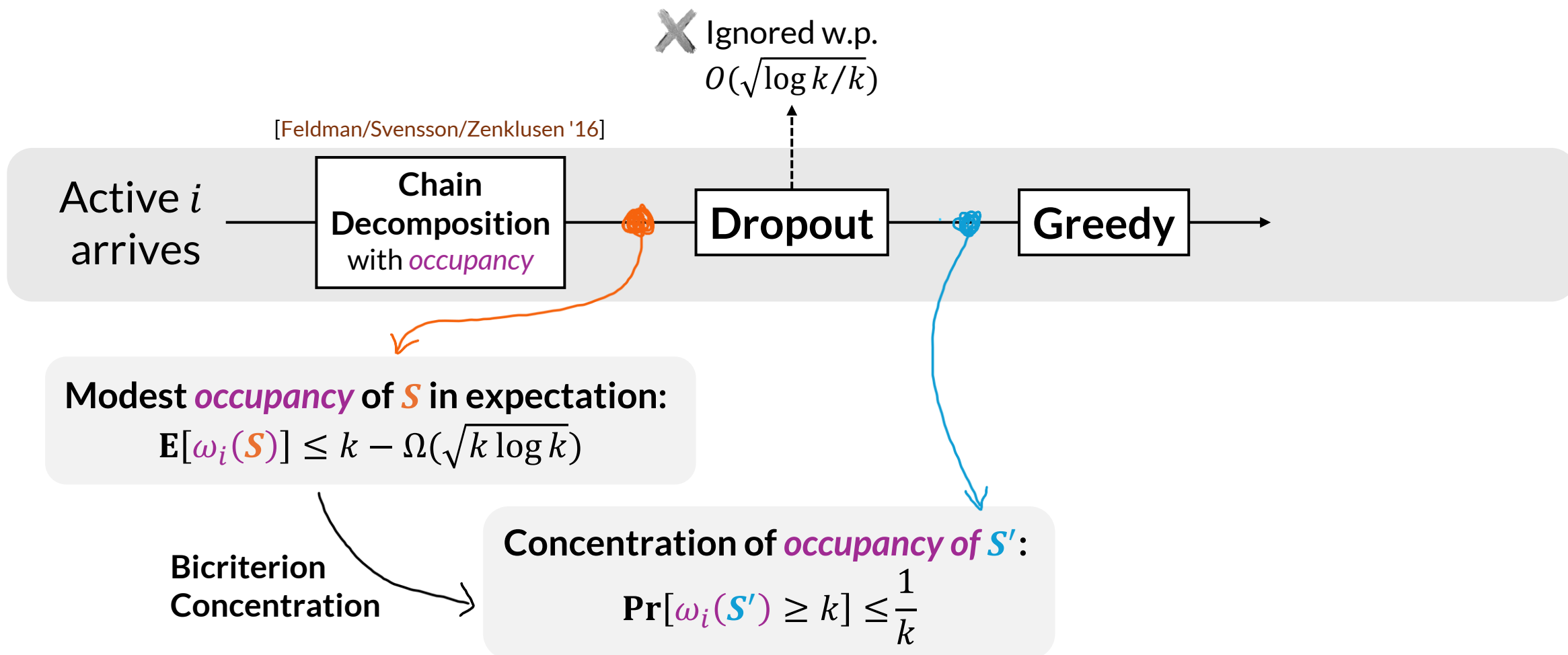
Modest *occupancy* of  $S$  in expectation:

$$\mathbb{E}[\omega_i(S)] \leq k - \Omega(\sqrt{k \log k})$$

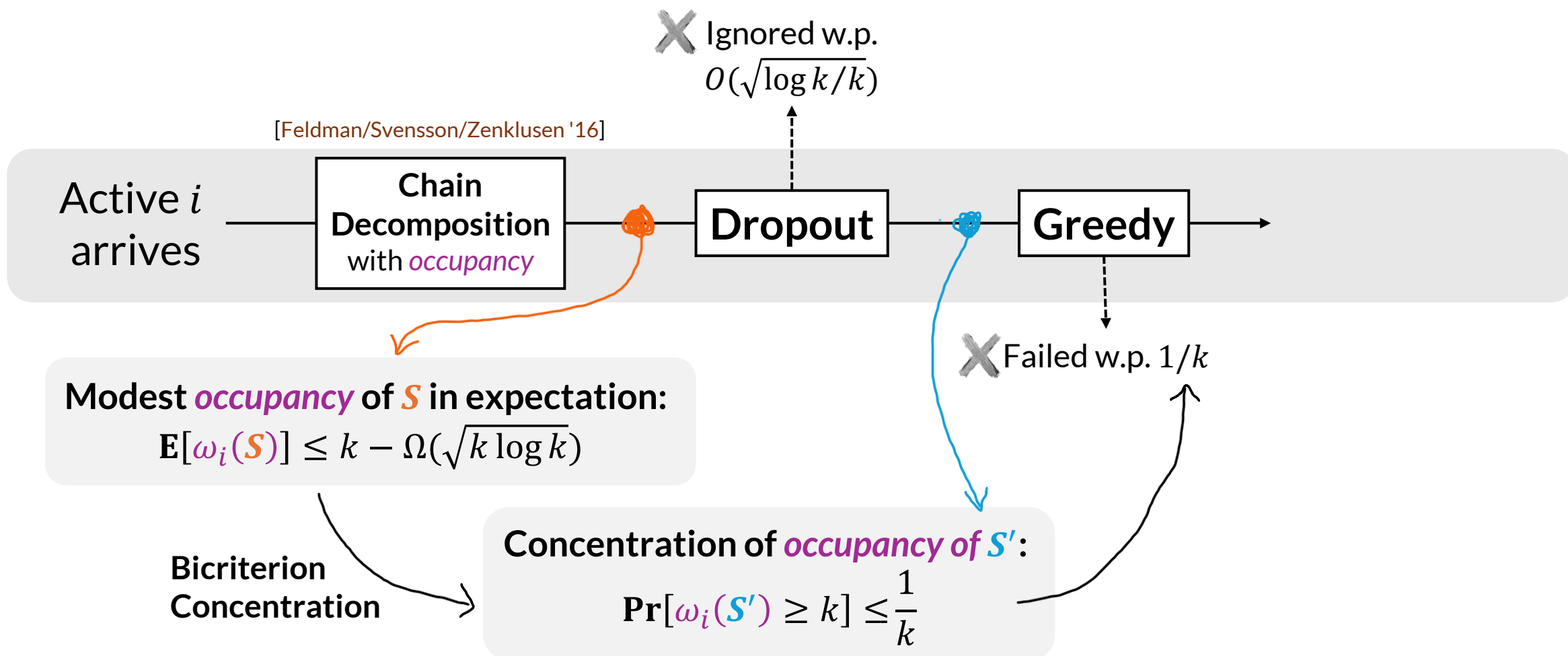
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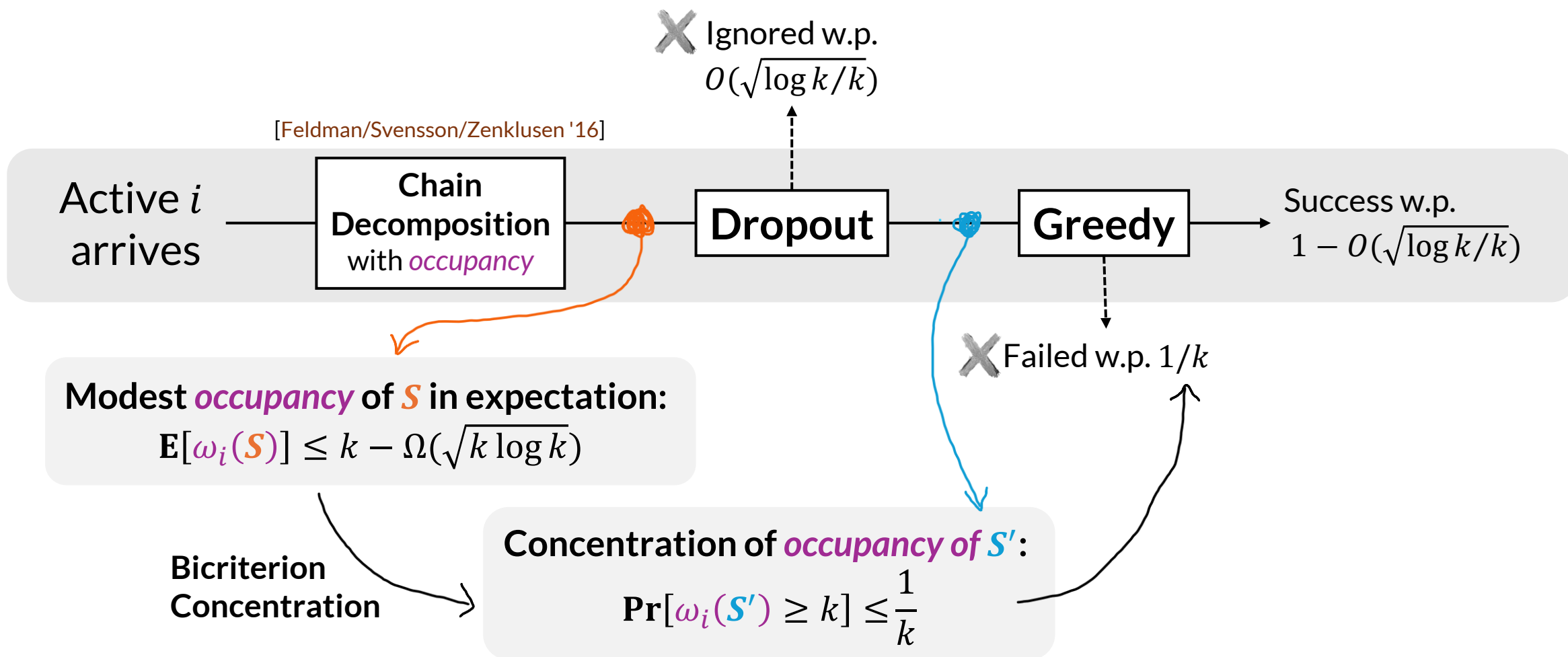
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# Conclusion

**Theorem:**  $\forall s \in [0,1], t > 0$

$$\Pr[f(\mathbf{X}^{(s)}) \geq \mathbf{E}[f(\mathbf{X})] + t] \leq e^{-st}$$

“Chernoff-strength” *bicriterion* concentration

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Improve to  $(1 - O(\frac{1}{\sqrt{k}}))$

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Thank you!