A *Bicriterion* **Concentration Inequality** and **Prophet Inequalities for** ***k-Fold Matroid Unions***

To appear at ITCS 2025

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Outline

Theorem: $\forall s \in [0,1], t > 0$

$$\mathbf{Pr}\left[f\left(\boldsymbol{X}^{(s)}\right) \geq \mathbf{E}[f(\boldsymbol{X})] + t\right] \leq e^{-st}$$

"Chernoff-strength" ***bicriterion*** concentration

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Large girth does not suffice for $(1 - \varepsilon)$-prophet inequality

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Theorem: There is $(1 - O(\sqrt{\frac{\log k}{k}}))$-competitive algorithm for any **$\boldsymbol{k}$-fold matroid union $\boldsymbol{\mathcal{F}^k}$**

But **$\boldsymbol{k}$-fold matroid unions** do

Part I:

A *Bicriterion* Concentration Inequality and Prophet Inequalities for *k-Fold Matroid Unions*

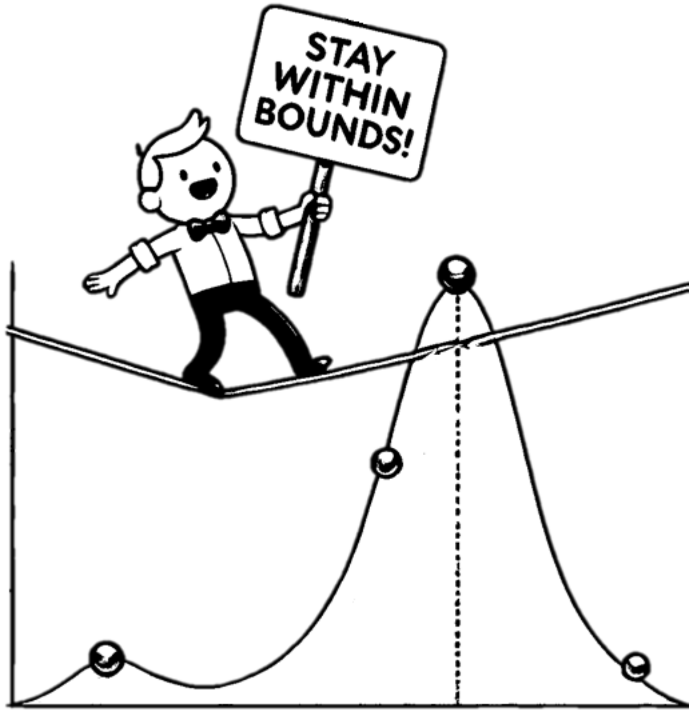

Image generated by ChatGPT

Concentration inequalities

n independent
Bernoulli r.v.s
$\boldsymbol{X} = (X_1, X_2, \dots, X_n)$

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$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq \|\boldsymbol{x} - \boldsymbol{y}\|_1$

"Flipping a single input bit change output by ≤ 1."

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How well does $f(\boldsymbol{X})$ concentrate around $\mu = \mathbf{E}[f(\boldsymbol{X})]$?

Focus on **upper tail** & **small deviation** $t < \mu$
$\mathbf{Pr}[f(\boldsymbol{X}) \ge \mu + t] \le ?$

Example: Chernoff bound

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$\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$

$$f(\boldsymbol{x}) = \sum_{i \in [n]} x_i$$

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(for $t < \mu$)

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Implies standard deviation

$\sigma = O(\sqrt{\mu})$

Example: submodular functions

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$\boldsymbol{X} = (X_1, X_2, \dots, X_n)$

$f : \{0,1\}^n \to \mathbb{R}$ that is

- Monotone **submodular**
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e.g., matroid rank functions

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Concentration for ***self-bounding functions***

[Boucheron/Lugosi/Massart '00]

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$$\mathbf{Pr}[f(\boldsymbol{X}) \geq \mu + t] \leq e^{-\frac{2t^2}{n}}$$

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In general, impossible to get ***"Chernoff-strength"*** bound

$\exists$ 1-Lipschitz f such that $f(\boldsymbol{X})$:

- (Small expectation) $\mu \ll \sqrt{n}$
- (Large deviation) $\sigma = \sqrt{n}$

Result 0: a *bicriterion* concentration

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"Scale down by $s \geq 0$":
set each X_i to 0 w.p. $1 - e^{-s}$

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dimension-free!

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$\boldsymbol{X}^{(s)} = \left(X_1^{(s)}, X_2^{(s)}, \dots, X_n^{(s)}\right)$

<u>Corollary</u> ("Chernoff-strength"): $\forall \delta \in [0,1]$,

$$\mathbf{Pr}\left[f\left(\boldsymbol{X}^{(\delta)}\right) \geq (1+\delta)\mathbf{E}[f(\boldsymbol{X})]\right] \leq e^{-\delta^2 \mathbf{E}[f(\boldsymbol{X})]}$$

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Intuition: for $f(\boldsymbol{x})$, either:
- Changes **fast**:
  "Scaling-down" to $f(\boldsymbol{X}^{(s)})$ helps
- Changes **slow:**
  $f(\boldsymbol{X})$ already concentrates

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n independent Bernoulli r.v.s

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Proof: entropy method with

$$F(\lambda) = \mathbf{E}\left[e^{-\lambda f(\boldsymbol{X}^{(\lambda)})}\right]$$

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Part II:

A *Bicriterion* Concentration Inequality and
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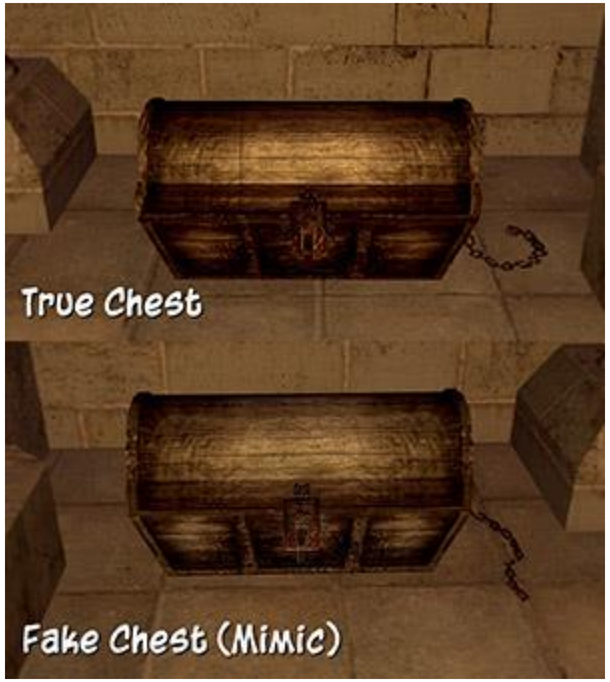
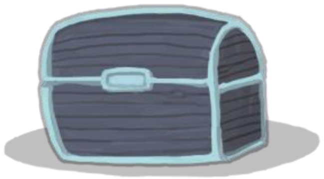
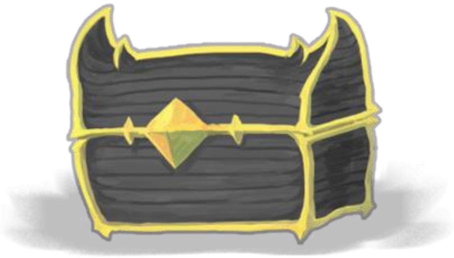


Image from http://darksouls.wikidot.com/mimic

Prophet inequalities

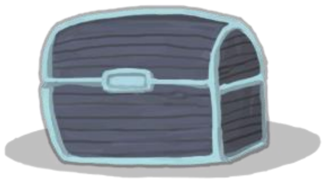

Images from “Slay the Spire”

Prophet inequalities

- Given n **independent** distributions $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$

$v_1 \sim \mathcal{D}_1$

$v_2 \sim \mathcal{D}_2$



$v_3 \sim \mathcal{D}_3$

$v_4 \sim \mathcal{D}_4$

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- At each step $i = 1,2, \dots, n$
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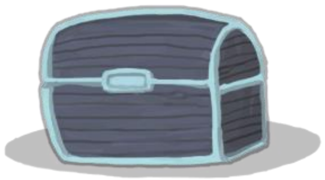

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Accept at most 1 item

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$v_1 = 6$

$v_2 \sim \mathcal{D}_2$

$v_3 \sim \mathcal{D}_3$

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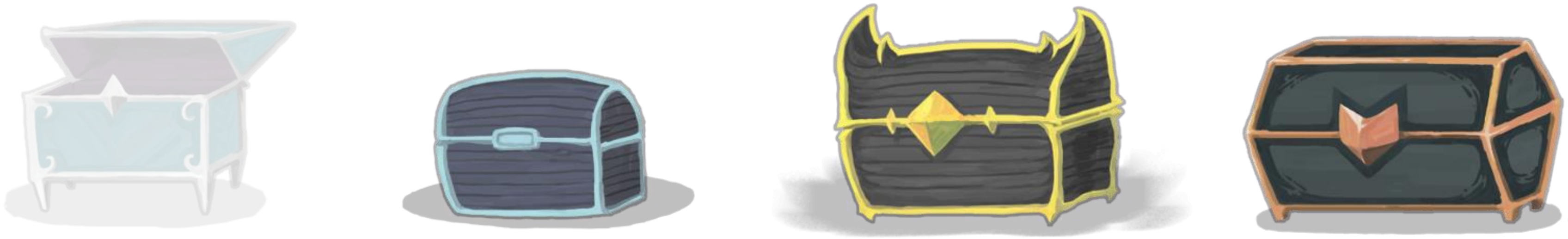

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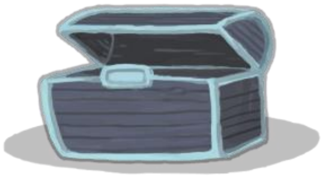

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Rejected $v_1 = 6$

Rejected $v_2 = 1$

$v_3 = 5$

$v_4 \sim \mathcal{D}_4$

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- Goal: **maximize** accepted value in expectation
  - vs. a *prophet* who gets $\mathbf{E}[\max_i v_i]$

Accept at most 1 item

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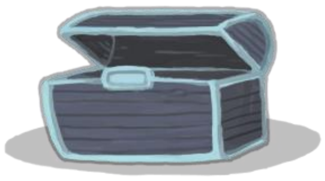

Accepted!
$v_3 = 5$

Prophet's value
$v_4 = 8$

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α-competitive:
$$\frac{\mathbf{E}[\mathrm{ALG}]}{\mathbf{E}[\mathrm{Prophet}]} \geq \alpha$$

Rejected
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Rejected
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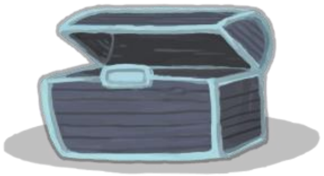

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$\frac{1}{2}$-competitive strategy: [Krengel/Sucheston/Garling '78, Samuel-Cahn '84]

$$\text{Accept first } v_i > T = \mathbf{Median}[\max_i v_i]$$

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Tight in worst case

Prophet inequalities (general feasibility)

- Given n independent $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$, **feasible sets $\mathcal{F} \subseteq \mathbf{2}^{[n]}$**
- At each step $i = 1,2, \dots, n$
  - Inspect $v_i \sim D_i$
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- Because of a *large girth*?
- Because it is a ***union of many matroids***?

Result 1: *large girth* does not suffice

Girth$(\mathcal{F})$: minimum size of infeasible set

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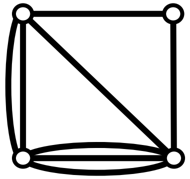
e.g.: graphical matroid of 

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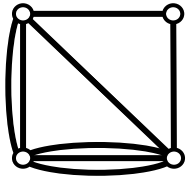
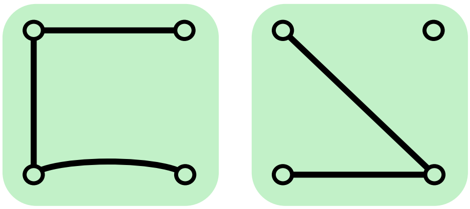
e.g.: graphical matroid of 



Feasible

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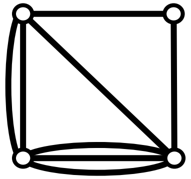

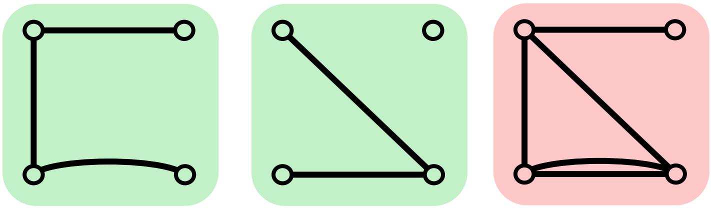

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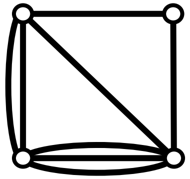
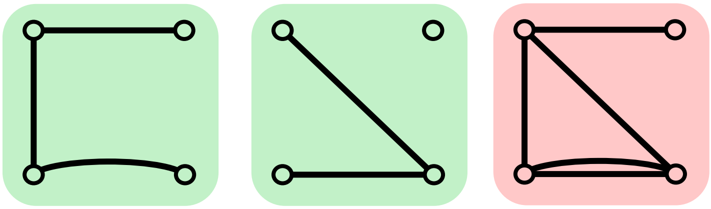
e.g.: graphical matroid of 



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<u>Theorem:</u> $\forall k, \varepsilon$, no $(\frac{1}{2}+\varepsilon)$-competitive algorithm for a graphical matroid $\mathcal{F}_{k,\varepsilon}$ of girth k

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***k*-fold union** of matroid $\mathcal{F}$:

Feasible $S \in \mathcal{F}^k$ $\Leftrightarrow$ partitioned into k feasible sets $\in \mathcal{F}$

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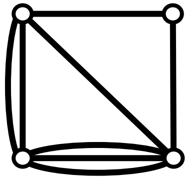

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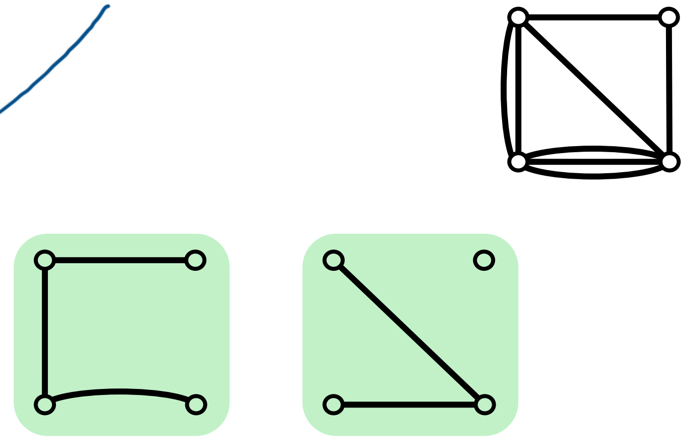

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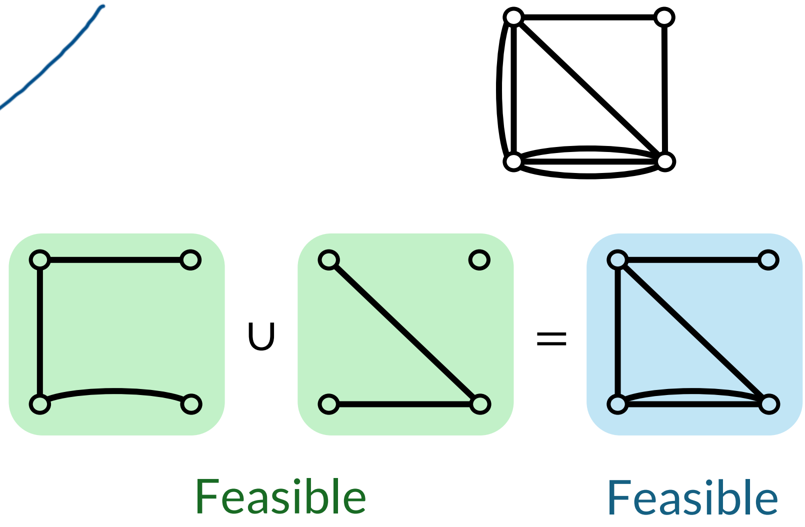

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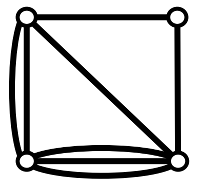

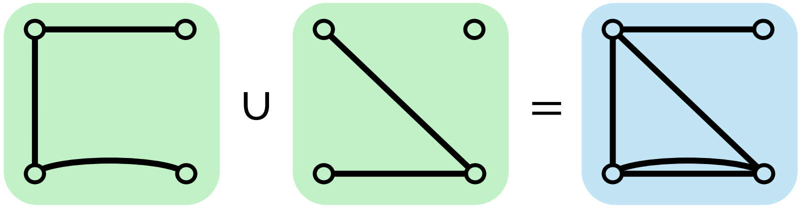

Feasible

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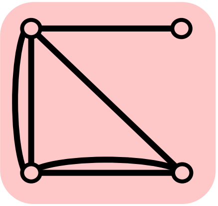

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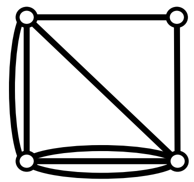

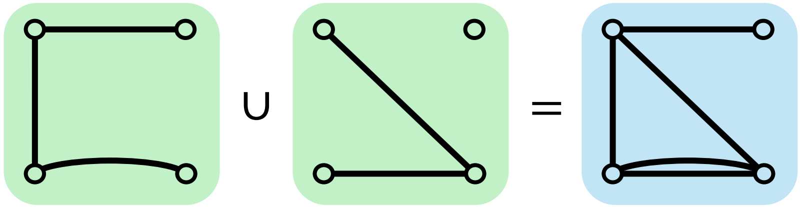

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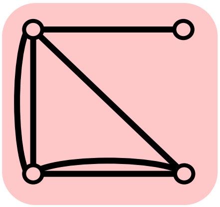

Infeasible

Theorem: There is $(1 - O(\sqrt{\frac{\log k}{k}}))$-competitive algorithm for any k-fold matroid union $\mathcal{F}^k$

Algorithm for k-uniform matroids

Call i active whenever $v_i > T_i$

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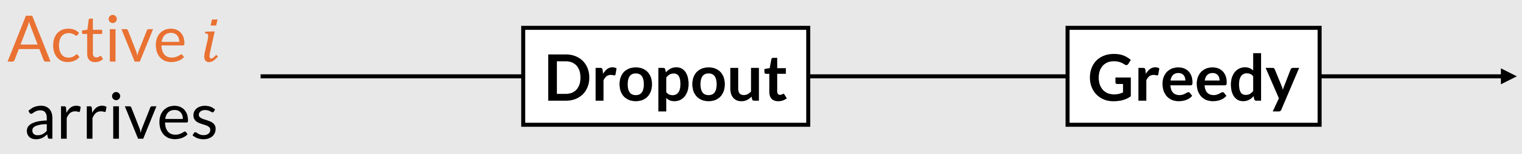

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Active i arrives $\longrightarrow$ **Dropout** $\longrightarrow$ **Greedy** $\longrightarrow$ Success w.p. $(1-\varepsilon)$

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$(1 - \varepsilon)$-competitive ratio!

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Failed w.p. ? ? ?

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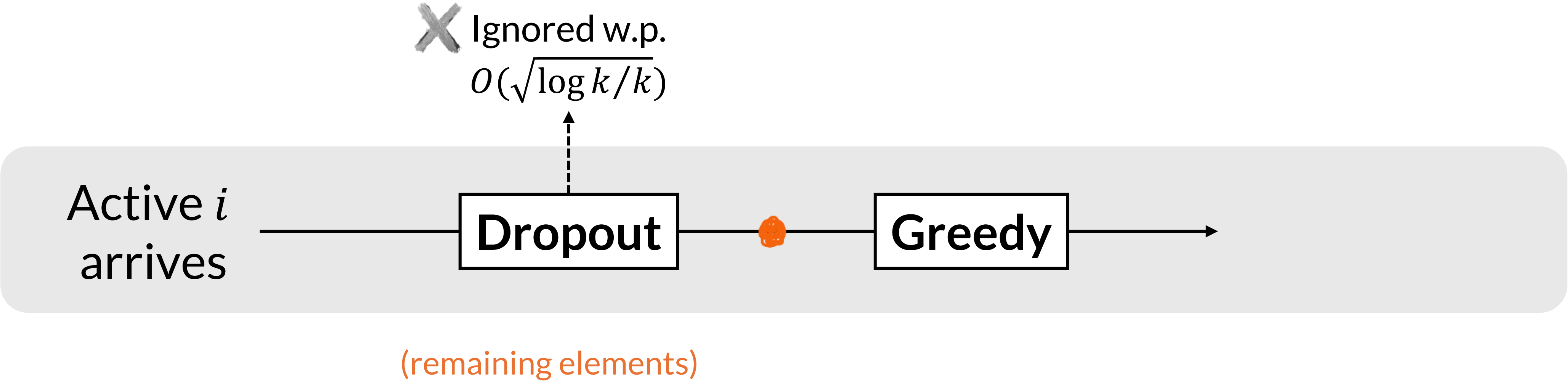

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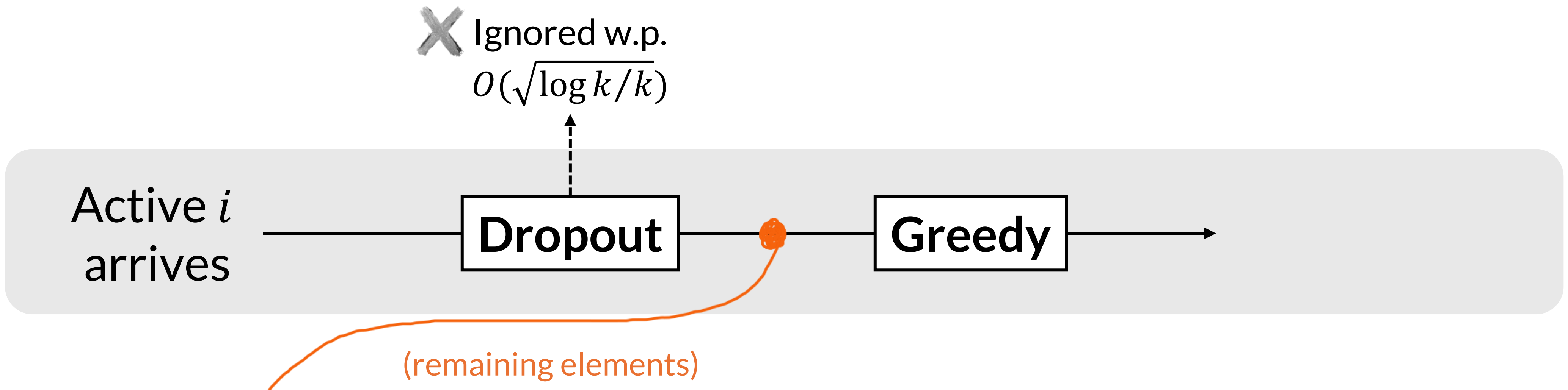
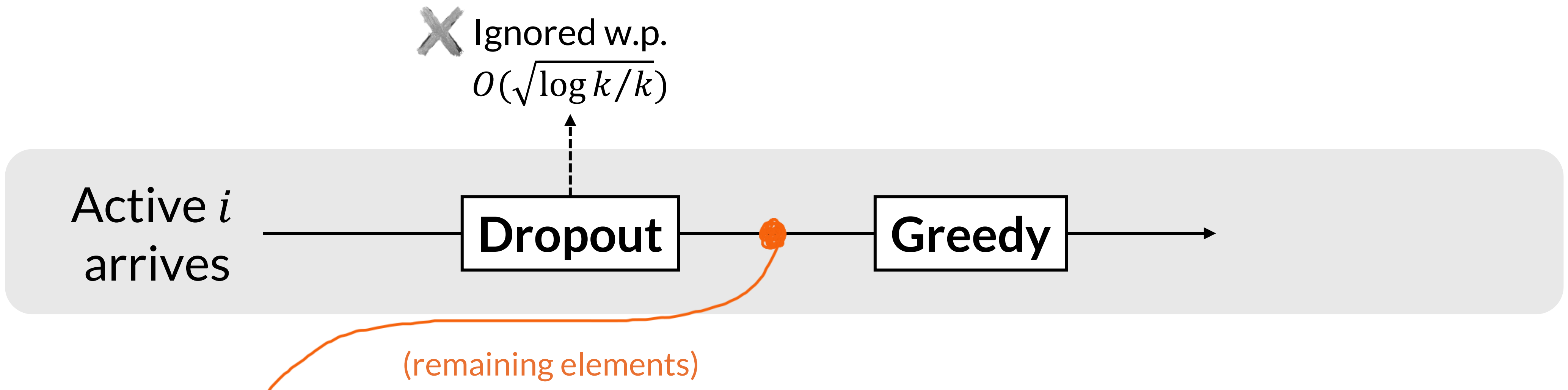


Modest *occupancy of S* in expectation:

$$\mathbf{E}[|S|] \leq k - \Omega(\sqrt{k \log k})$$

Algorithm for k-uniform matroids



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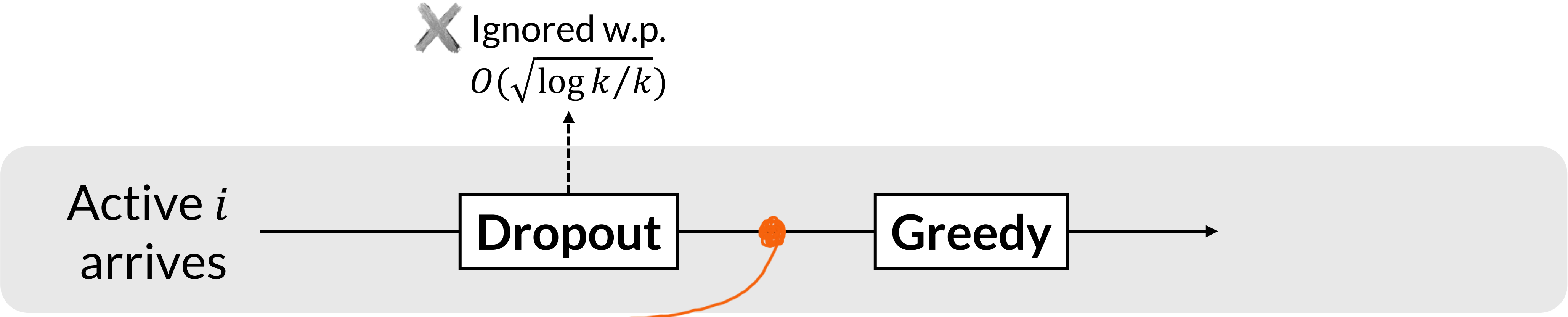

(remaining elements)

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Chernoff bound

Concentration of *occupancy*:

$$\mathbf{Pr}\Big[|S| \geq \mathbf{E}[|S|] + O(\sqrt{k \log k})\Big] \leq \frac{1}{k}$$

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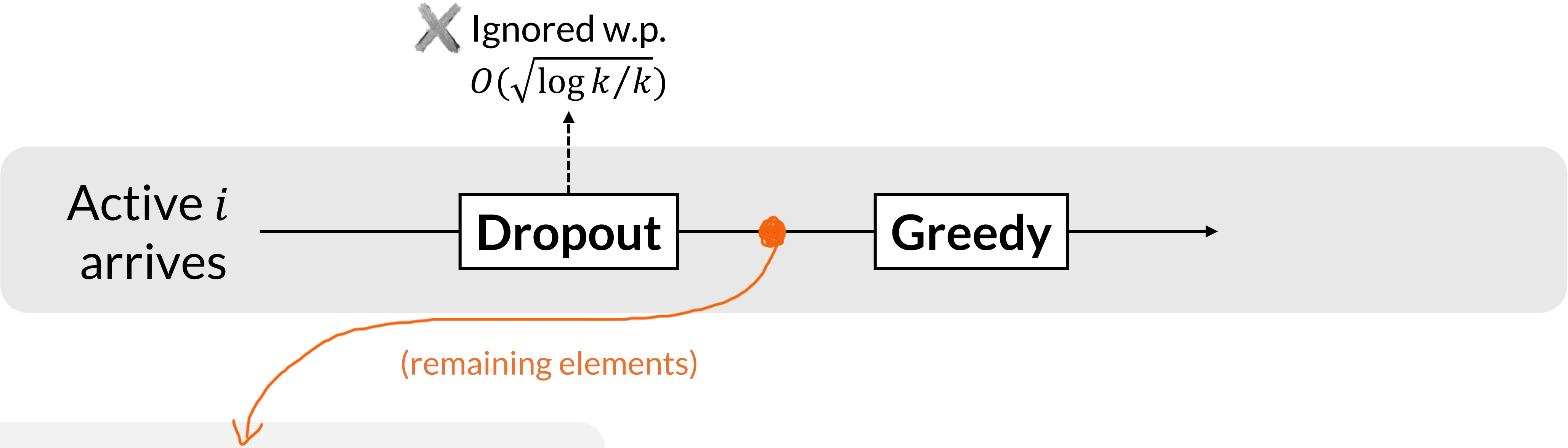

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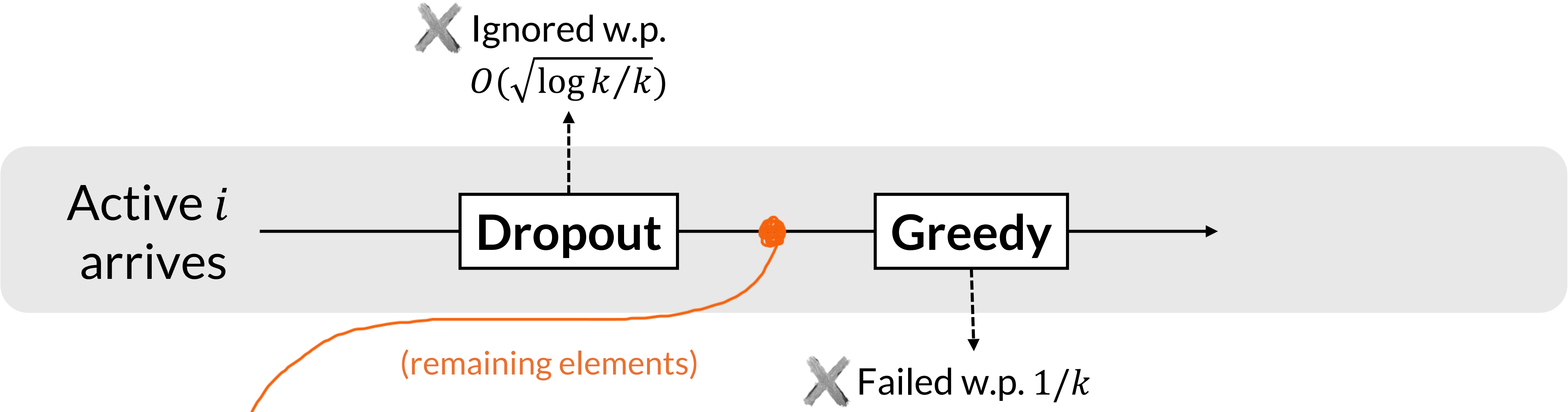

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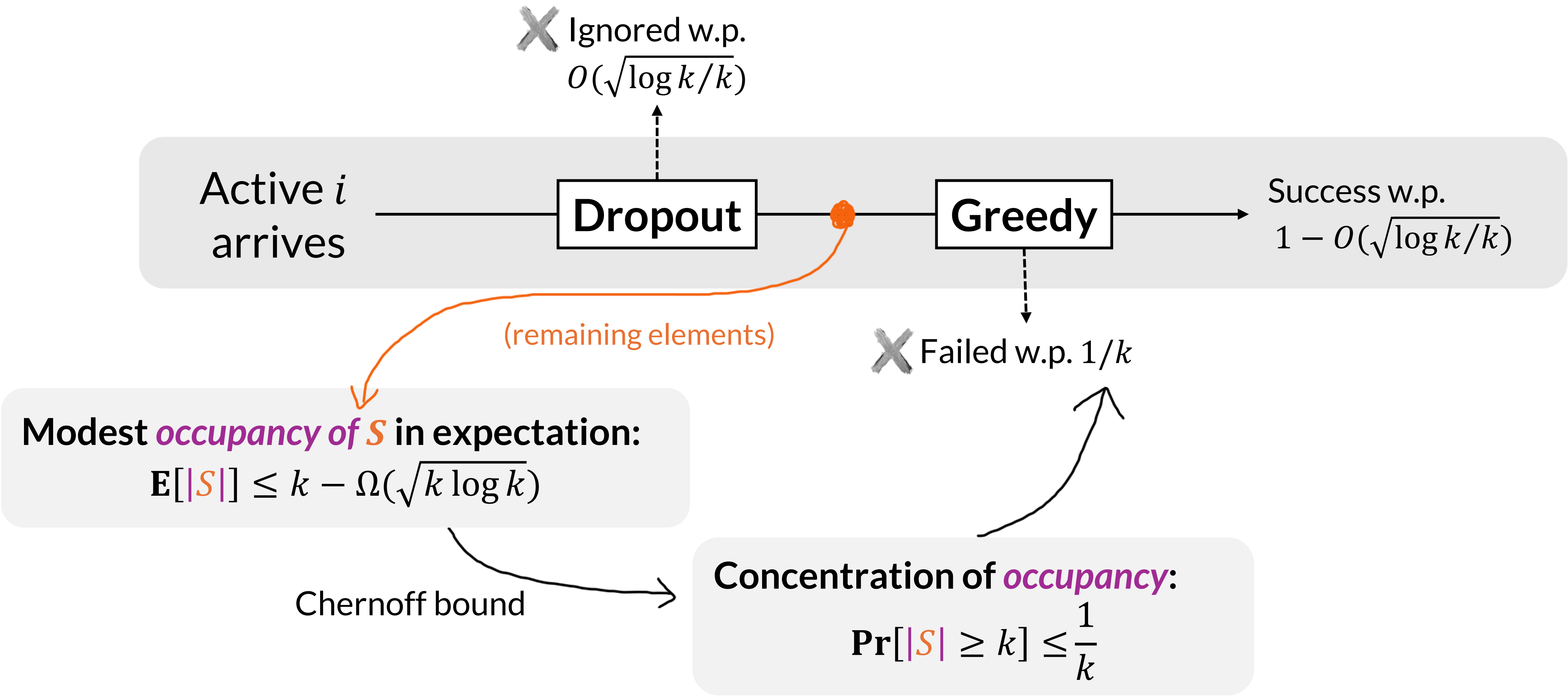

Algorithm for k-uniform matroids

Ignored w.p. $O(\sqrt{\log k / k})$

Active i arrives → **Dropout** → **Greedy** → Success w.p. $1 - O(\sqrt{\log k / k})$

(remaining elements)

Failed w.p. $1/k$

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Generalize to ***k*-fold matroid union?**

Concentration of *occupancy*:

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Occupancy in k-fold matroid unions

Occupancy: "slots" occupied by S

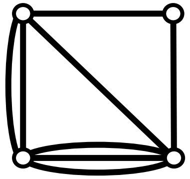
Occupancy in k-fold matroid unions

Occupancy: "slots" occupied by S with respect to **specific element**

Occupancy in k-fold matroid unions

Occupancy: “slots” occupied by S with respect to **specific element**

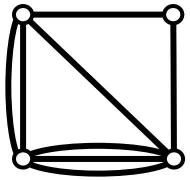
In 2-fold union of



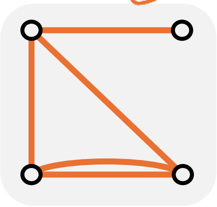
Occupancy in k-fold matroid unions

Occupancy: “slots” occupied by S
with respect to **specific element**

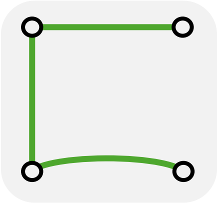
In 2-fold union of



Occupancy of

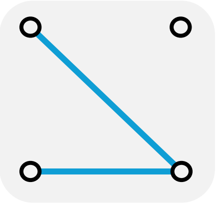


w.r.t.



Slot 1

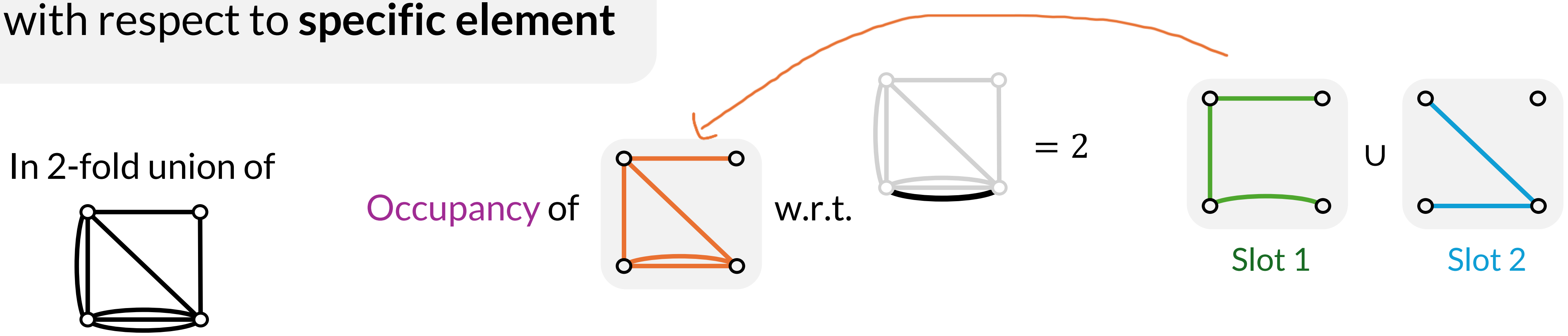
∪



Slot 2

Occupancy in k-fold matroid unions

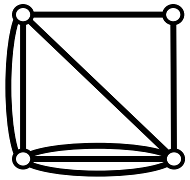
Occupancy: "slots" occupied by S with respect to **specific element**



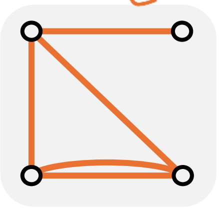
Occupancy in k-fold matroid unions

Occupancy: "slots" occupied by S
with respect to **specific element**

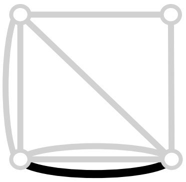
In 2-fold union of



Occupancy of

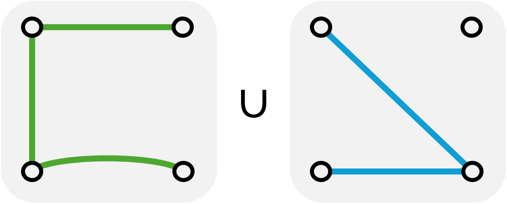


w.r.t.



$= 2$

$= 1$



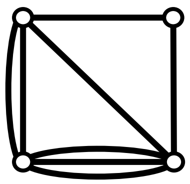
Slot 1

Slot 2

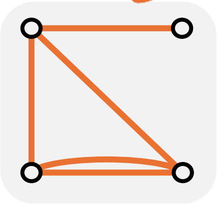
Occupancy in k-fold matroid unions

Occupancy: “slots” occupied by S with respect to **specific element**

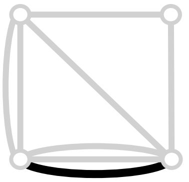
In 2-fold union of



Occupancy of



w.r.t.



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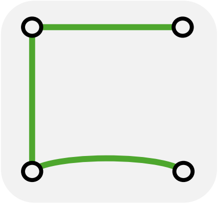
$= 1$



Slot 1

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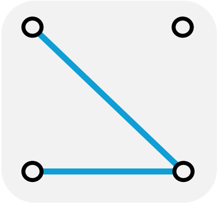


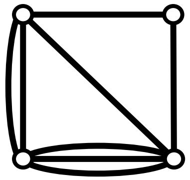
Slot 2

Lemma: there exists *occupancy function* ω_i for every i:

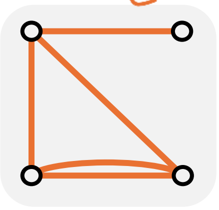
Occupancy in k-fold matroid unions

Occupancy: “slots” occupied by S
with respect to **specific element**

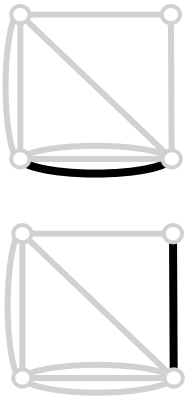
In 2-fold union of



Occupancy of

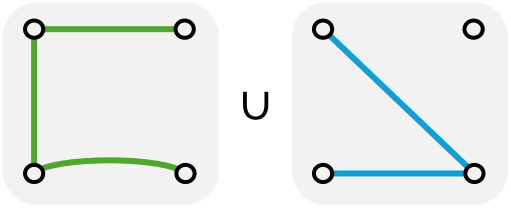


w.r.t.



$= 2$

$= 1$



$\cup$

Slot 1 Slot 2

<u>Lemma</u>: there exists *occupancy function* ω_i for every i:

1. Maps every $S \subseteq [n]$ to integer $\{0,1,2,\ldots,k\}$

Occupancy in k-fold matroid unions

Occupancy: "slots" occupied by S
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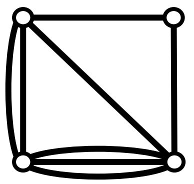
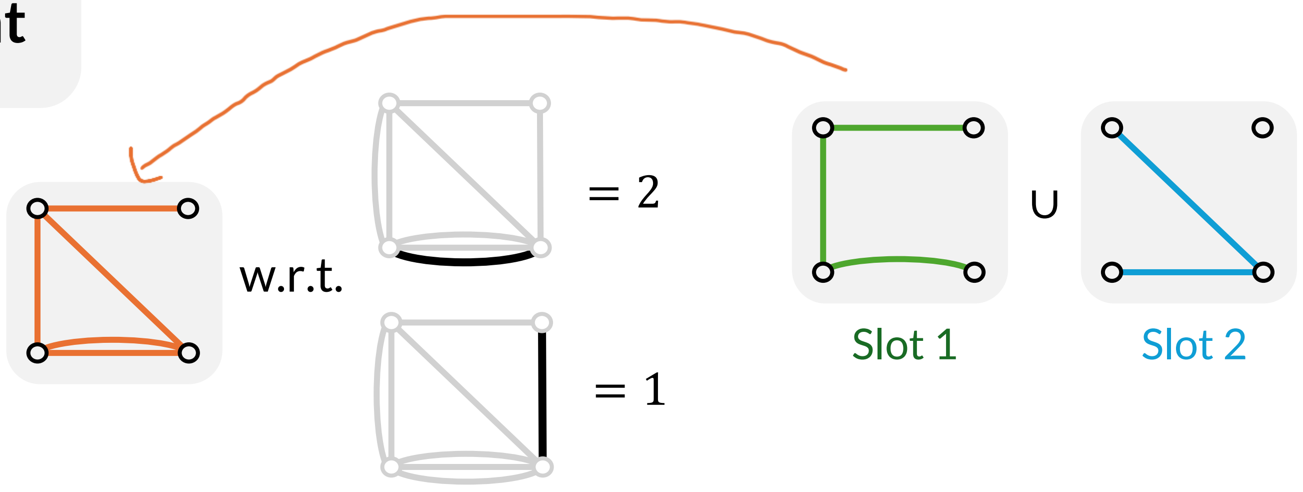
In 2-fold union of



Occupancy of



<u>Lemma</u>: there exists *occupancy function* ω_i for every i:

1. Maps every $S \subseteq [n]$ to integer $\{0,1,2,\ldots,k\}$
2. $\omega_i(S) < k \;\Rightarrow\; i$ is "compatible" with S

Occupancy in k-fold matroid unions

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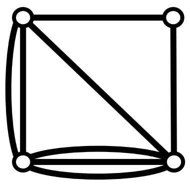
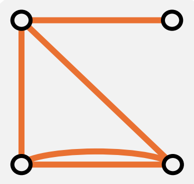
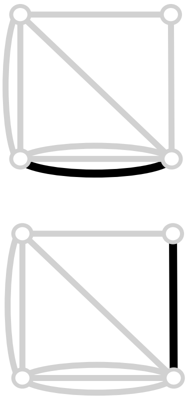
In 2-fold union of



Occupancy of

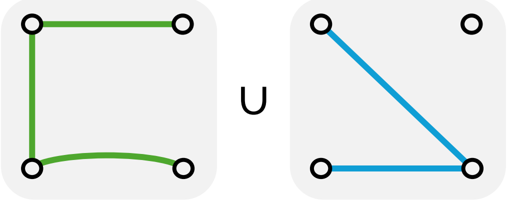


w.r.t.



$= 2$

$= 1$



$\cup$

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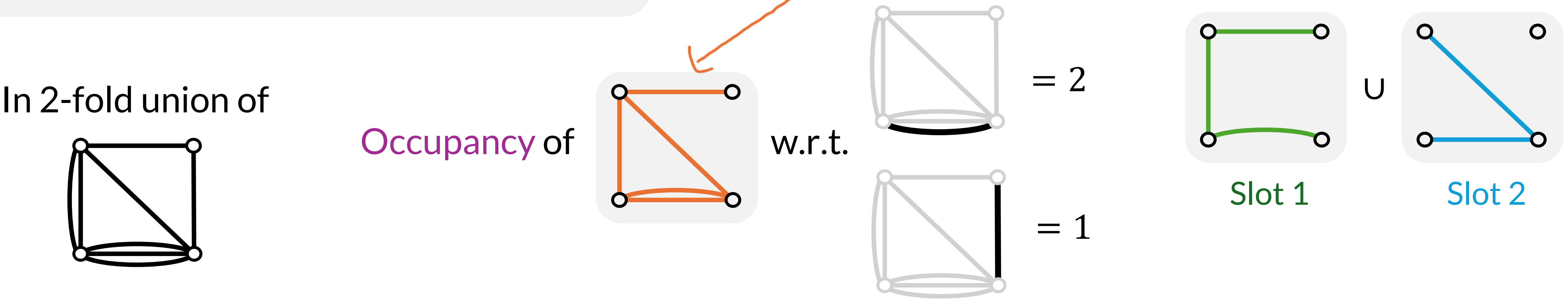
i can be added to
any feasible subset of S

Occupancy in k-fold matroid unions

Occupancy: "slots" occupied by S
with respect to **specific element**



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3. Monotone and 1-Lipschitz

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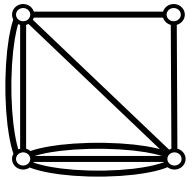
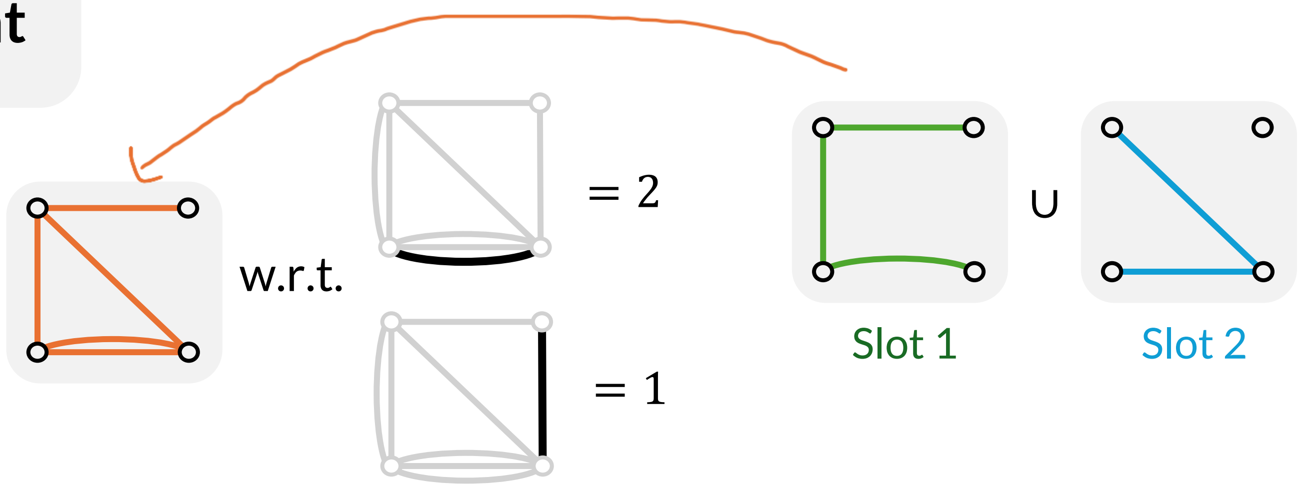
In 2-fold union of



Occupancy of



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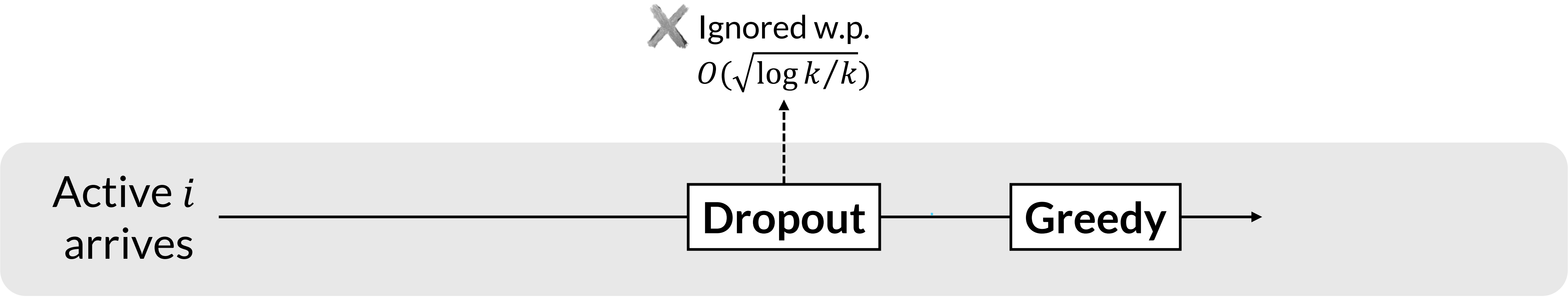
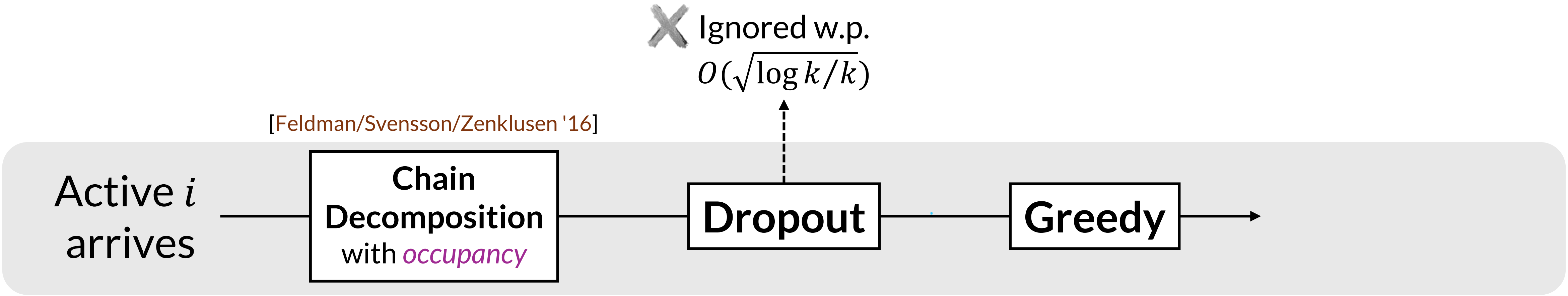
i can be added to
any feasible subset of S

Can be very general
No hope for Chernoff-like bounds

Algorithm for k-fold matroid unions



Algorithm for k-fold matroid unions



Algorithm for k-fold matroid unions

Ignored w.p. $O(\sqrt{\log k / k})$

[Feldman/Svensson/Zenklusen '16]

Active i arrives

Chain Decomposition with *occupancy*

Dropout

Greedy

Modest *occupancy* of S in expectation:

$$\mathbf{E}[\omega_i(S)] \leq k - \Omega(\sqrt{k \log k})$$

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Bicriterion Concentration

Concentration of *occupancy of S'*:

$$\mathbf{Pr}\left[\omega_i(S') \geq \mathbf{E}[\omega_i(S)] + \sqrt{k \log k}\right] \leq \frac{1}{k}$$

Algorithm for k-fold matroid unions

Ignored w.p.
$O(\sqrt{\log k / k})$

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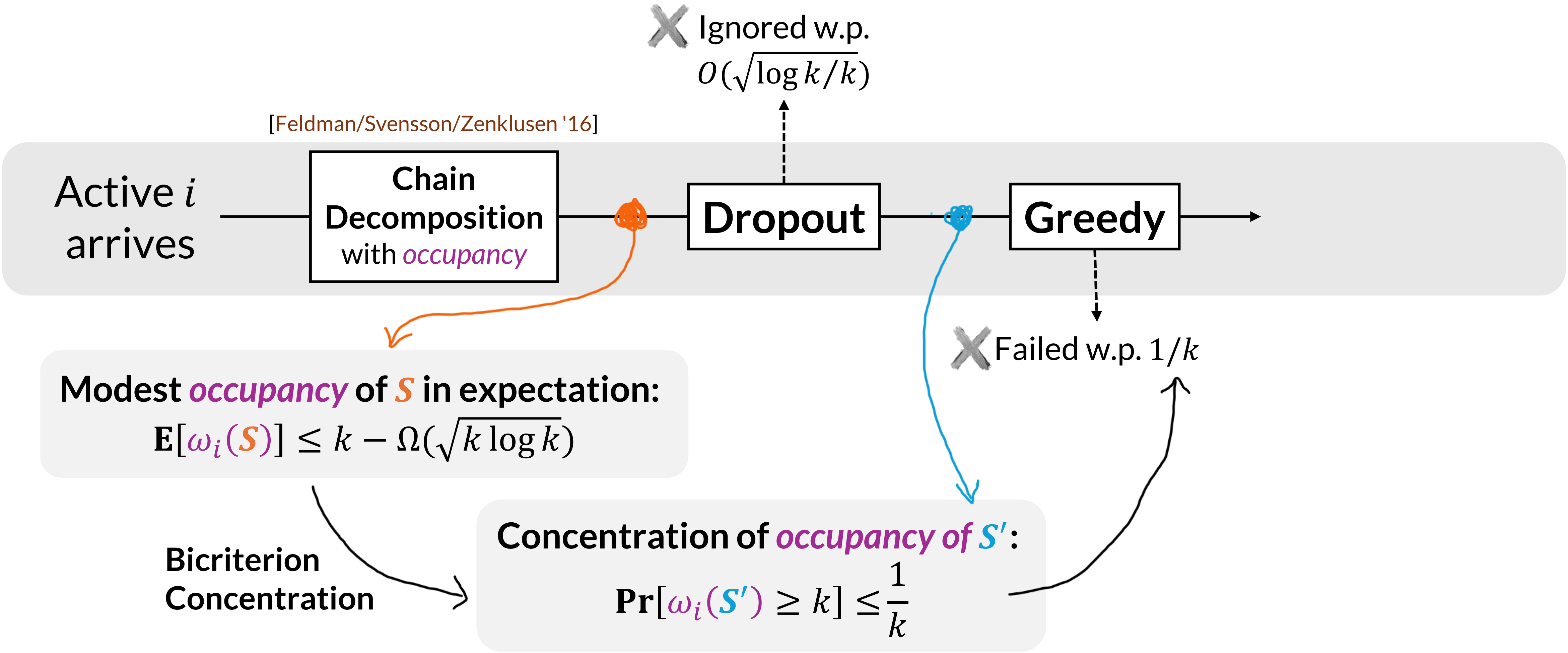
Bicriterion Concentration

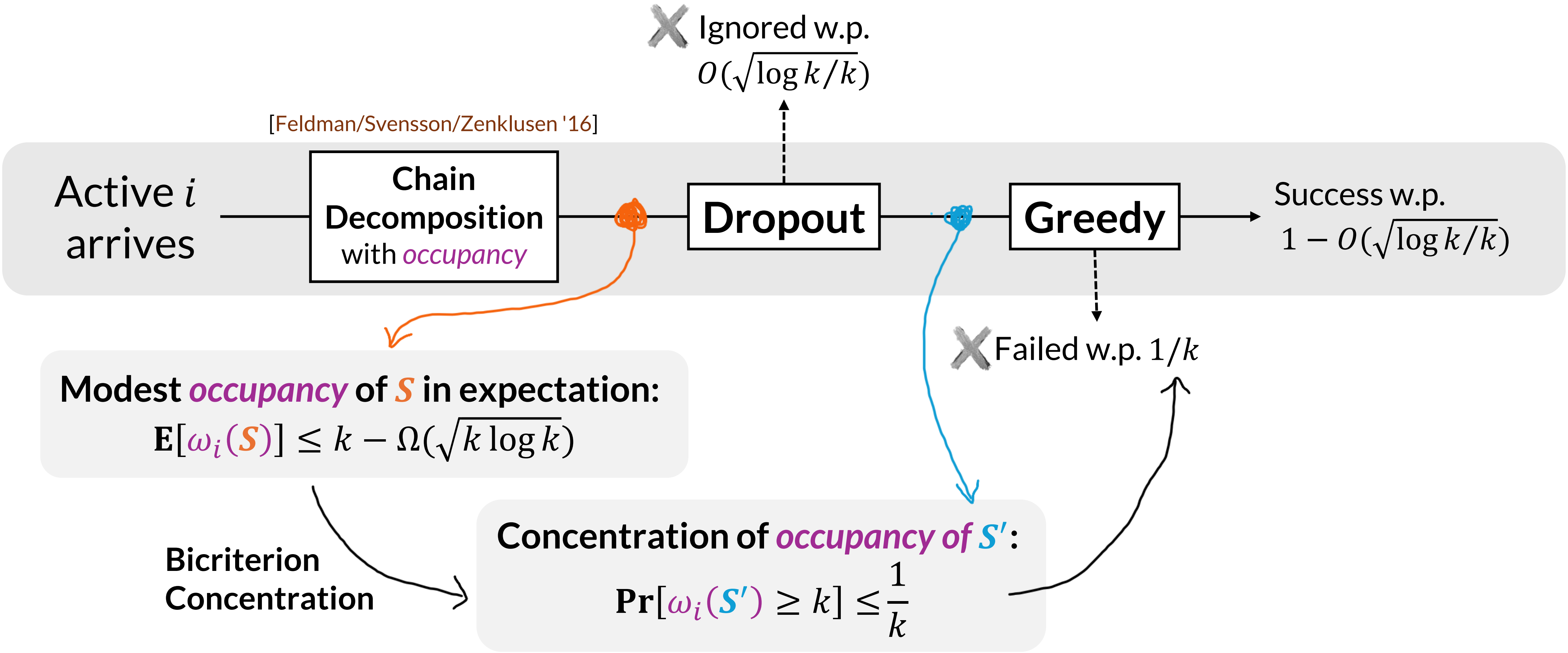
Concentration of *occupancy of* S':

$$\mathbf{Pr}[\omega_i(S') \geq k] \leq \frac{1}{k}$$

Algorithm for k-fold matroid unions



Algorithm for k-fold matroid unions



Conclusion

Theorem: $\forall s \in [0,1], t > 0$

$$\mathbf{Pr}\left[f\left(\boldsymbol{X}^{(s)}\right) \geq \mathbf{E}[f(\boldsymbol{X})] + t\right] \leq e^{-st}$$

"Chernoff-strength" ***bicriterion*** concentration

Conclusion

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Theorem: $\forall k, \varepsilon$, no $(\frac{1}{2} + \varepsilon)$-competitive algorithm for a graphical matroid $\mathcal{F}_{k,\varepsilon}$ of **girth $\boldsymbol{k}$**

Large girth does not suffice for $(1 - \varepsilon)$-prophet inequality

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But **k-fold matroid unions** do

Conclusion

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Open question 1:
Improve to $(1 - O(\frac{1}{\sqrt{k}}))$

Conclusion

Theorem: $\forall s \in [0,1], t > 0$
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"Chernoff-strength" ***bicriterion*** concentration

Theorem: $\forall k, \varepsilon$, no $(\frac{1}{2} + \varepsilon)$-competitive algorithm for a graphical matroid $\mathcal{F}_{k,\varepsilon}$ of **girth k**

Large girth does not suffice for $(1-\varepsilon)$-prophet inequality

Theorem: There is $(1 - O(\sqrt{\frac{\log k}{k}}))$-competitive algorithm for any **k-fold matroid union $\mathcal{F}^k$**

But **k-fold matroid unions** do

Open question 2:
Beyond $\mathcal{F}^k$?

Open question 1:
Improve to $(1 - O(\frac{1}{\sqrt{k}}))$

Conclusion

Theorem: $\forall s \in [0,1], t > 0$
$\mathbf{Pr}\left[f\left(X^{(s)}\right) \geq \mathbf{E}[f(X)] + t\right] \leq e^{-st}$

"Chernoff-strength" ***bicriterion*** concentration

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Thank you!